Worked Examples in Vector Mechanics

Geometry, Kinematics, Dynamics

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Chapter 1

Vectors

1.1 Distribution

Distribution is a useful method for calculating dot products and cross products. Calculating a dot product or cross product by distribution consists of three steps:

1. **Distribute terms:** Note the <u>order</u> of terms must be preserved.

2. **Combine scalars:** Scalar multiplication with vectors is associative.

3. **Use definitions:** Use the definition of dot product and cross product as necessary.

Example:

Calculate the following dot and cross products. Assume $\hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_y$, $\hat{\mathbf{n}}_z$ form a right-handed, orthonormal basis.

$$
\begin{aligned}
5\,\hat{\mathbf{n}}_x \cdot (2\,\hat{\mathbf{n}}_x + 3\,\hat{\mathbf{n}}_y) &= 5\,\hat{\mathbf{n}}_x \cdot 2\,\hat{\mathbf{n}}_x + 5\,\hat{\mathbf{n}}_x \cdot 3\,\hat{\mathbf{n}}_y && \text{Distribute terms, preserve order.}\\
&= 10\,(\hat{\mathbf{n}}_x \cdot \hat{\mathbf{n}}_x) + 15\,(\hat{\mathbf{n}}_x \cdot \hat{\mathbf{n}}_y) && \text{Combine scalars.}\\
&= 10\,(1) \qquad\ + 15\,(0) && \text{Use definition of dot product.}\\
&= 10
\end{aligned}
$$

$$
\begin{aligned}
5\,\hat{\mathbf{n}}_x \times (2\,\hat{\mathbf{n}}_x + 3\,\hat{\mathbf{n}}_y) &= 5\,\hat{\mathbf{n}}_x \times 2\,\hat{\mathbf{n}}_x + 5\,\hat{\mathbf{n}}_x \times 3\,\hat{\mathbf{n}}_y && \text{Distribute terms, preserve order.}\\
&= 10\,(\hat{\mathbf{n}}_x \times \hat{\mathbf{n}}_x) + 15\,(\hat{\mathbf{n}}_x \times \hat{\mathbf{n}}_y) && \text{Collect scalars.}\\
&= 10\,(\vec{\mathbf{0}}) \qquad\ + 15\,(\hat{\mathbf{n}}_z) && \text{Use definition of cross product.}\\
&= 15\,\hat{\mathbf{n}}_z
\end{aligned}
$$

Example:

Consider arbitrary vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ (*arbitrary* means they are not necessarily unit-length or orthogonal to each other). Show that $(4\,\vec{\mathbf{u}} - 7\,\vec{\mathbf{w}}) \cdot 3\,\vec{\mathbf{u}} \times 2\,\vec{\mathbf{v}}$ simplifies to $(42\,\vec{\mathbf{w}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{u}}))$.

Solution:

In more complex problems like this, the steps outlined above may be used multiple times or in a different order.

$$
\begin{aligned}
&(4\,\vec{\mathbf{u}} - 7\,\vec{\mathbf{w}}) \cdot 3\,\vec{\mathbf{u}} \times 2\,\vec{\mathbf{v}} &&\\
={}& (4\,\vec{\mathbf{u}} - 7\,\vec{\mathbf{w}}) \cdot (3\,\vec{\mathbf{u}} \times 2\,\vec{\mathbf{v}}) && \text{Cross product must occur before dot product.}\\
={}& (4\,\vec{\mathbf{u}} - 7\,\vec{\mathbf{w}}) \cdot (6\,\vec{\mathbf{u}} \times \vec{\mathbf{v}}) && \text{Collect scalars in cross product.}\\
={}& 4\,\vec{\mathbf{u}} \cdot (6\,\vec{\mathbf{u}} \times \vec{\mathbf{v}}) - 7\,\vec{\mathbf{w}} \cdot (6\,\vec{\mathbf{u}} \times \vec{\mathbf{v}}) && \text{Can't simplify } \vec{\mathbf{u}} \times \vec{\mathbf{v}}, \text{ so treat as a single vector quantity and distribute.}\\
={}& 24\,\vec{\mathbf{u}} \cdot (\vec{\mathbf{u}} \times \vec{\mathbf{v}}) - 42\,\vec{\mathbf{w}} \cdot (\vec{\mathbf{u}} \times \vec{\mathbf{v}}) && \text{Collect scalars in dot products.}\\
={}& 24\,(0) - 42\,\vec{\mathbf{w}} \cdot (\vec{\mathbf{u}} \times \vec{\mathbf{v}}) && \vec{\mathbf{u}} \times \vec{\mathbf{v}} \text{ is perpendicular to both } \vec{\mathbf{u}} \text{ and } \vec{\mathbf{v}}, \text{ so } \vec{\mathbf{u}} \cdot (\vec{\mathbf{u}} \times \vec{\mathbf{v}}) = 0.\\
={}& {-42}\,\vec{\mathbf{w}} \cdot (\vec{\mathbf{u}} \times \vec{\mathbf{v}}) && \text{Reverse cross product since } (\vec{\mathbf{u}} \times \vec{\mathbf{v}}) = -(\vec{\mathbf{v}} \times \vec{\mathbf{u}}).\\
={}& 42\,\vec{\mathbf{w}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{u}}) && \text{Cannot simplify these arbitrary vectors any further.}
\end{aligned}
$$

1.1.1 Organizing Your Work

When distributing a dot or cross product for two vectors with many terms it is easy to omit or mix-up terms. These errors can be avoided by carefully organizing your work (as described below) so that errors stand out visually.

Consider the vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$. To calculate $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}$ or $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$, organize the distribution as follows (see the examples below):

1. Distribute the first term of vector $\vec{\mathbf{a}}$ to each of the terms of vector $\vec{\mathbf{b}}$, preserving the ordering.

2. Go to the next line.

3. Distribute the next term of vector $\vec{\mathbf{a}}$ to each of the terms of vector $\vec{\mathbf{b}}$, preserving the ordering.

4. Repeat step 2 and 3 for each of the terms in vector $\vec{\mathbf{a}}$.

This will naturally organize the expanded terms into a shape where the number of rows is equal to the number of terms in vector $\vec{\mathbf{a}}$ and a number of columns is equal to the number of terms in vector $\vec{\mathbf{b}}$. This layout makes it easy to visually inspect the rows and columns of the result for any mistakes.[1]

Example:

Consider the force $\vec{\mathbf{F}} = F_1\,\widehat{\mathbf{n}}_x + F_2\,\widehat{\mathbf{n}}_y$ and the velocity of the point where $\vec{\mathbf{F}}$ is applied, $\vec{\mathbf{v}} = \dot{x}\,\widehat{\mathbf{n}}_x + r\,\dot{\theta}\,\widehat{\mathbf{b}}_x + \dot{z}\,\widehat{\mathbf{n}}_z$. Calculate the power of the force, defined as $P \triangleq \vec{\mathbf{F}} \cdot \vec{\mathbf{v}}$. Assume $\widehat{\mathbf{n}}_x, \widehat{\mathbf{n}}_y, \widehat{\mathbf{n}}_z$ and $\widehat{\mathbf{b}}_x, \widehat{\mathbf{b}}_y, \widehat{\mathbf{b}}_z$ are each right-handed, orthonormal bases, however the relationship between $\widehat{\mathbf{n}}_{xyz}$ and $\widehat{\mathbf{b}}_{xyz}$ is not specified.

Solution:

$$P \triangleq \vec{\mathbf{F}} \cdot \vec{\mathbf{v}} = (F_1\,\widehat{\mathbf{n}}_x + F_2\,\widehat{\mathbf{n}}_y) \cdot (\dot{x}\,\widehat{\mathbf{n}}_x + r\,\dot{\theta}\,\widehat{\mathbf{b}}_x + \dot{z}\,\widehat{\mathbf{n}}_z) \qquad \text{Setup}$$

$$= \quad F_1\,\widehat{\mathbf{n}}_x \cdot \dot{x}\,\widehat{\mathbf{n}}_x + F_1\,\widehat{\mathbf{n}}_x \cdot r\,\dot{\theta}\,\widehat{\mathbf{b}}_x + F_1\,\widehat{\mathbf{n}}_x \cdot \dot{z}\,\widehat{\mathbf{n}}_z \quad \text{Distribute first term across second grouping.}$$

$$+ F_2\,\widehat{\mathbf{n}}_y \cdot \dot{x}\,\widehat{\mathbf{n}}_x + F_2\,\widehat{\mathbf{n}}_y \cdot r\,\dot{\theta}\,\widehat{\mathbf{b}}_x + F_2\,\widehat{\mathbf{n}}_y \cdot \dot{z}\,\widehat{\mathbf{n}}_z \quad \text{Distribute second term across second grouping.}$$

Notice the distinct pattern. First, we can check that the dimensions of the result (2 rows by 3 columns) match the number of terms in the vectors $\vec{\mathbf{F}}$ and $\vec{\mathbf{v}}$. Next, we see the terms in the first vector ($\vec{\mathbf{F}}$) are written in order down each of the columns, and come first in each of the dot products. The terms in the second vector ($\vec{\mathbf{v}}$) are written in order across each of the rows, and come second in each of the dot products. By writing the expanded result in this shape it is easy to check that no terms were omitted, that the ordering was preserved everywhere, and that no copying errors were made (e.g. accidentally writing $\widehat{\mathbf{n}}_x$ where there should be $\widehat{\mathbf{b}}_x$).

$$= \quad F_1\,\dot{x}\,(\widehat{\mathbf{n}}_x \cdot \widehat{\mathbf{n}}_x) + F_1\,r\,\dot{\theta}\,(\widehat{\mathbf{n}}_x \cdot \widehat{\mathbf{b}}_x) + F_1\,\dot{z}\,(\widehat{\mathbf{n}}_x \cdot \widehat{\mathbf{n}}_z) \quad \text{Collect Scalars}$$

$$+ F_2\,\dot{x}\,(\widehat{\mathbf{n}}_y \cdot \widehat{\mathbf{n}}_x) + F_2\,r\,\dot{\theta}\,(\widehat{\mathbf{n}}_y \cdot \widehat{\mathbf{b}}_x) + F_2\,\dot{z}\,(\widehat{\mathbf{n}}_y \cdot \widehat{\mathbf{n}}_z)$$

$$= \quad F_1\,\dot{x}\,(1) + F_1\,r\,\dot{\theta}\,(\widehat{\mathbf{n}}_x \cdot \widehat{\mathbf{b}}_x) + F_1\,\dot{z}\,(0) \qquad \text{Apply Definitions}$$

$$+ F_2\,\dot{x}\,(0) + F_2\,r\,\dot{\theta}\,(\widehat{\mathbf{n}}_y \cdot \widehat{\mathbf{b}}_x) + F_2\,\dot{z}\,(0)$$

$$= F_1\,\dot{x} + F_1\,r\,\dot{\theta}\,(\widehat{\mathbf{n}}_x \cdot \widehat{\mathbf{b}}_x) + F_2\,r\,\dot{\theta}\,(\widehat{\mathbf{n}}_y \cdot \widehat{\mathbf{b}}_x) \qquad \text{Simplify}$$

In a problem with a picture (or if a rotation table were given) the relationship between $\widehat{\mathbf{n}}_x, \widehat{\mathbf{n}}_y, \widehat{\mathbf{n}}_z$ and $\widehat{\mathbf{b}}_x, \widehat{\mathbf{b}}_y, \widehat{\mathbf{b}}_z$ could be deduced, and hence the dot products $\widehat{\mathbf{n}}_x \cdot \widehat{\mathbf{b}}_x$ and $\widehat{\mathbf{n}}_y \cdot \widehat{\mathbf{b}}_x$ could be found. However, without further information in this example we cannot simplify this result any further.

1.2 Shortcuts for Calculating Certain Dot Products and Cross Products

In introductory physics or math classes it is common to only use a single right-handed, orthonormal vector basis, such as $\widehat{\mathbf{i}}, \widehat{\mathbf{j}}, \widehat{\mathbf{k}}$ or $\widehat{\mathbf{n}}_x, \widehat{\mathbf{n}}_y, \widehat{\mathbf{n}}_z$. When calculating the dot product or cross product of two vectors both expressed in terms of the same right-handed, orthonormal basis it becomes possible to use some convenient shortcuts for calculation. While these methods are useful, it must be stressed that they are **special-case** formulas and **not** definitions.
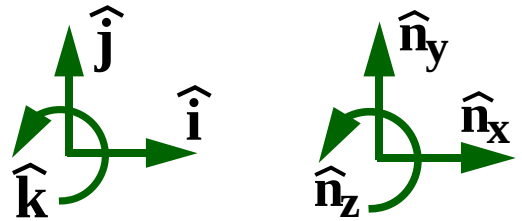
1.2.1 Calculating Dot Product in a Single Orthonormal Basis

The familiar result that dot product is calculated as the sum of products of components is a **special case**. To see how this arises, consider the right-handed, orthonormal basis $\widehat{\mathbf{n}}_x, \widehat{\mathbf{n}}_y, \widehat{\mathbf{n}}_z$ and the dot product $\vec{\mathbf{u}} \cdot \vec{\mathbf{w}}$, with $\vec{\mathbf{u}} = a_1\,\widehat{\mathbf{n}}_x + a_2\,\widehat{\mathbf{n}}_y + a_3\,\widehat{\mathbf{n}}_z$ and $\vec{\mathbf{w}} = b_1\,\widehat{\mathbf{n}}_x + b_2\,\widehat{\mathbf{n}}_y + b_3\,\widehat{\mathbf{n}}_z$.

---

[1] Those familiar with matrix algebra will recognize this looks similar to an *outer-product*, if the terms of $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ were organized into column-matrix form.

Then we have:

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{w}} = (a_1\,\widehat{\mathbf{n}}_x \,+\, a_2\,\widehat{\mathbf{n}}_y \,+\, a_3\,\widehat{\mathbf{n}}_z) \cdot (b_1\,\widehat{\mathbf{n}}_x \,+\, b_2\,\widehat{\mathbf{n}}_y \,+\, b_3\,\widehat{\mathbf{n}}_z) \qquad \text{Setup}$$

$$
\begin{aligned}
= \quad & a_1\,b_1\,(\widehat{\mathbf{n}}_x\cdot\widehat{\mathbf{n}}_x) && + \; a_1\,b_2\,(\widehat{\mathbf{n}}_x\cdot\widehat{\mathbf{n}}_y) && + \; a_1\,b_3\,(\widehat{\mathbf{n}}_x\cdot\widehat{\mathbf{n}}_z) && \text{Distribute and Collect Scalars}\\
& + a_2\,b_1\,(\widehat{\mathbf{n}}_y\cdot\widehat{\mathbf{n}}_x) && + \; a_2\,b_2\,(\widehat{\mathbf{n}}_y\cdot\widehat{\mathbf{n}}_y) && + \; a_2\,b_3\,(\widehat{\mathbf{n}}_y\cdot\widehat{\mathbf{n}}_z) \\
& + a_3\,b_1\,(\widehat{\mathbf{n}}_z\cdot\widehat{\mathbf{n}}_x) && + \; a_3\,b_2\,(\widehat{\mathbf{n}}_z\cdot\widehat{\mathbf{n}}_y) && + \; a_3\,b_3\,(\widehat{\mathbf{n}}_z\cdot\widehat{\mathbf{n}}_z)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & a_1\,b_1\,(1) && + \; a_1\,b_2\,(0) && + \; a_1\,b_3\,(0) && \text{Apply Definitions}\\
& + a_2\,b_1\,(0) && + \; a_2\,b_2\,(1) && + \; a_2\,b_3\,(0) \\
& + a_3\,b_1\,(0) && + \; a_3\,b_2\,(0) && + \; a_3\,b_3\,(1)
\end{aligned}
$$

$$= a_1\,b_1 \,+\, a_2\,b_2 \,+\, a_3\,b_3 \qquad\qquad\qquad \text{Simplify}$$

We can see that in the **special case** where two vectors are expressed in the same right-handed, orthonormal basis, then the dot product <u>may</u> be calculated as the sum of the products of the components:

$$(a_1\,\widehat{\mathbf{n}}_x \,+\, a_2\,\widehat{\mathbf{n}}_y \,+\, a_3\,\widehat{\mathbf{n}}_z) \cdot (b_1\,\widehat{\mathbf{n}}_x \,+\, b_2\,\widehat{\mathbf{n}}_y \,+\, b_3\,\widehat{\mathbf{n}}_z) \overset{(\text{special})}{=} a_1\,b_1 \,+\, a_2\,b_2 \,+\, a_3\,b_3 \tag{1.1}$$

Example:

The **velocity** of a particle is $\vec{\mathbf{v}} = v_x\,\widehat{\mathbf{b}}_x \,+\, v_y\,\widehat{\mathbf{b}}_y \,+\, 0\,\widehat{\mathbf{b}}_z$, where $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$ form a right-handed, orthonormal basis. Calculate the particle's **kinetic energy**, defined as $K \triangleq \frac{1}{2}\,m\,\vec{\mathbf{v}}\cdot\vec{\mathbf{v}}$.
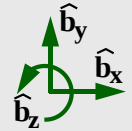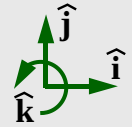
Solution:

Since the vector $\vec{\mathbf{v}}$ is expressed in terms of a **single** right-handed, orthonormal basis, equation 1.1 applies.

$$K \triangleq \tfrac{1}{2}\,m\,\vec{\mathbf{v}}\cdot\vec{\mathbf{v}} \;=\; \tfrac{1}{2}\,m\,(v_x\,\widehat{\mathbf{b}}_x \,+\, v_y\,\widehat{\mathbf{b}}_y \,+\, 0\,\widehat{\mathbf{b}}_z)\cdot(v_x\,\widehat{\mathbf{b}}_x \,+\, v_y\,\widehat{\mathbf{b}}_y \,+\, 0\,\widehat{\mathbf{b}}_z) \;=\; \tfrac{1}{2}\,m\,(v_x^2 + v_y^2 + 0)$$

Example:

The **position vector** of a point Q from a point P is $\vec{\mathbf{r}} = 2\,\widehat{\mathbf{i}} \,+\, 7\,\widehat{\mathbf{j}} \,-\, 3\,\widehat{\mathbf{k}}$, where $\widehat{\mathbf{i}}$, $\widehat{\mathbf{j}}$, $\widehat{\mathbf{k}}$ form a right-handed, orthonormal basis. Find the distance, d, between Q and P by calculating $|\vec{\mathbf{r}}|$, the magnitude of the position vector.

Solution:

Since the vector $\vec{\mathbf{r}}$ is expressed in terms of a **single** right-handed, orthornormal basis, equation 1.1 applies.

$$d = \left|{}^{P}\vec{\mathbf{r}}^{\,Q}\right| = \sqrt{{}^{P}\vec{\mathbf{r}}^{\,Q}\cdot{}^{P}\vec{\mathbf{r}}^{\,Q}} = \sqrt{(2\,\widehat{\mathbf{i}}+7\,\widehat{\mathbf{j}}-3\,\widehat{\mathbf{k}})\cdot(2\,\widehat{\mathbf{i}}+7\,\widehat{\mathbf{j}}-3\,\widehat{\mathbf{k}})} \;=\; \sqrt{(2)(2)+(7)(7)+(-3)(-3)} \;=\; \sqrt{62}$$

1.2.2 Calculating Single-Basis Cross Products

Several shortcut methods exist for calculating the cross product of two vectors expressed in the same right-handed, orthonormal basis. To derive these **special-case** shortcuts, consider the right-handed, orthonormal basis $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$ and the cross product $\vec{\mathbf{u}}\times\vec{\mathbf{w}}$, with $\vec{\mathbf{u}} = a_1\,\widehat{\mathbf{n}}_x \,+\, a_2\,\widehat{\mathbf{n}}_y \,+\, a_3\,\widehat{\mathbf{n}}_z$ and $\vec{\mathbf{w}} = b_1\,\widehat{\mathbf{n}}_x \,+\, b_2\,\widehat{\mathbf{n}}_y \,+\, b_3\,\widehat{\mathbf{n}}_z$.

Then we have:

$$\vec{\mathbf{u}} \times \vec{\mathbf{w}} = (a_1\,\widehat{\mathbf{n}}_x \,+\, a_2\,\widehat{\mathbf{n}}_y \,+\, a_3\,\widehat{\mathbf{n}}_z) \times (b_1\,\widehat{\mathbf{n}}_x \,+\, b_2\,\widehat{\mathbf{n}}_y \,+\, b_3\,\widehat{\mathbf{n}}_z) \qquad \text{Setup}$$

$$
\begin{aligned}
= \quad & a_1\,b_1\,(\widehat{\mathbf{n}}_x\times\widehat{\mathbf{n}}_x) && + \; a_1\,b_2\,(\widehat{\mathbf{n}}_x\times\widehat{\mathbf{n}}_y) && + \; a_1\,b_3\,(\widehat{\mathbf{n}}_x\times\widehat{\mathbf{n}}_z) && \text{Distribute and Collect Scalars}\\
& + a_2\,b_1\,(\widehat{\mathbf{n}}_y\times\widehat{\mathbf{n}}_x) && + \; a_2\,b_2\,(\widehat{\mathbf{n}}_y\times\widehat{\mathbf{n}}_y) && + \; a_2\,b_3\,(\widehat{\mathbf{n}}_y\times\widehat{\mathbf{n}}_z) \\
& + a_3\,b_1\,(\widehat{\mathbf{n}}_z\times\widehat{\mathbf{n}}_x) && + \; a_3\,b_2\,(\widehat{\mathbf{n}}_z\times\widehat{\mathbf{n}}_y) && + \; a_3\,b_3\,(\widehat{\mathbf{n}}_z\times\widehat{\mathbf{n}}_z)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & a_1\,b_1\,(\vec{\mathbf{0}}) && + \; a_1\,b_2\,(\widehat{\mathbf{n}}_z) && + \; a_1\,b_3\,(-\widehat{\mathbf{n}}_y) && \text{Apply Definitions}\\
& + a_2\,b_1\,(-\widehat{\mathbf{n}}_z) && + \; a_2\,b_2\,(\vec{\mathbf{0}}) && + \; a_2\,b_3\,(\widehat{\mathbf{n}}_x) \\
& + a_3\,b_1\,(\widehat{\mathbf{n}}_y) && + \; a_3\,b_2\,(-\widehat{\mathbf{n}}_x) && + \; a_3\,b_3\,(\vec{\mathbf{0}})
\end{aligned}
$$

$$= (a_2\,b_3 - a_3\,b_2)\,\widehat{\mathbf{n}}_x \,+\, (a_3\,b_1 - a_1\,b_3)\,\widehat{\mathbf{n}}_y \,+\, (a_1\,b_2 - a_2\,b_1)\,\widehat{\mathbf{n}}_z \quad \text{Simplify}$$

We can see that in the **special case** where two vectors are expressed in the same right-handed, orthonormal basis, then their cross product <u>may</u> be calculated as:

$$(a_1 \ \widehat{\mathbf{n}}_x + a_2 \ \widehat{\mathbf{n}}_y + a_3 \ \widehat{\mathbf{n}}_z) \times (b_1 \ \widehat{\mathbf{n}}_x + b_2 \ \widehat{\mathbf{n}}_y + b_3 \ \widehat{\mathbf{n}}_z) \overset{\text{(special)}}{=} (a_2 \ b_3 - a_3 \ b_2) \ \widehat{\mathbf{n}}_x + (a_3 \ b_1 - a_1 \ b_3) \ \widehat{\mathbf{n}}_y + (a_1 \ b_2 - a_2 \ b_1) \ \widehat{\mathbf{n}}_z \quad (1.2)$$

Unlike the special-case dot product formula, the result for cross product is rather complex. Instead of memorizing this result we prefer to make use of one of the methods that follow.

Cross Product via Matrix Determinant

One method for arriving at the previous result is to arrange the basis vectors and coefficients in a 3x3 matrix and calculate the matrix determinant. To calculate the product $\vec{\mathbf{u}} \times \vec{\mathbf{w}}$ in the **special case** where $\vec{\mathbf{u}}$ and $\vec{\mathbf{w}}$ are expressed in terms of the same right-handed, orthonormal basis:

1. Write the basis vectors in the first row of the matrix, in some order.

2. Write the coefficients of $\vec{\mathbf{u}}$ in the second row, using the same order.

3. Write the coefficients of $\vec{\mathbf{w}}$ in the third row, using the same order.

$$\vec{\mathbf{u}} \times \vec{\mathbf{w}} \overset{\text{(special)}}{=} \det\left(\begin{bmatrix} \widehat{\mathbf{n}}_x & \widehat{\mathbf{n}}_y & \widehat{\mathbf{n}}_z \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \right)$$

4. Calculate the determinant of the matrix.

Some common methods for computing a 3×3 matrix determinant are (1) expansion by minors, and (2) the diagonal method. Details for executing these methods can be readily found through an internet search.

1.2.3 Cross Product Memory Devices for Unit Vectors in a Single Basis

Several methods can be used to memorize (or deduce) the cross products between the unit vectors in a single right-handed, orthonormal basis. These methods all rely on the common idea that the cross product of any two vectors of such a basis will be parallel to the third vector, subject to a + or - sign (unless crossing one of the basis vectors with itself, in which case the result is $\vec{\mathbf{0}}$ by definition).

Alphabetical Ordering

Perhaps the simplest method is to simply appeal to the alphabetical ordering of the unit vectors. Using the the sequence $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$ as an example, the sign is positive if the crossed vectors are in alphabetical order (e.g. $\widehat{\mathbf{n}}_x \times \widehat{\mathbf{n}}_y = \widehat{\mathbf{n}}_z$), and negative otherwise (e.g. $\widehat{\mathbf{n}}_y \times \widehat{\mathbf{n}}_x = -\widehat{\mathbf{n}}_z$). Note that the vector ordering has "wrap-around"; to help visualize this, it is may be useful to write the first two vectors in the sequence again at the end of the sequence (e.g. $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$, $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$), so that the two unit vectors in a cross product can always be found as an adjacent pair in the list (e.g. $\widehat{\mathbf{n}}_z \times \widehat{\mathbf{n}}_x = \widehat{\mathbf{n}}_y$).

For visual learners, the visual representations shown at right may be useful.
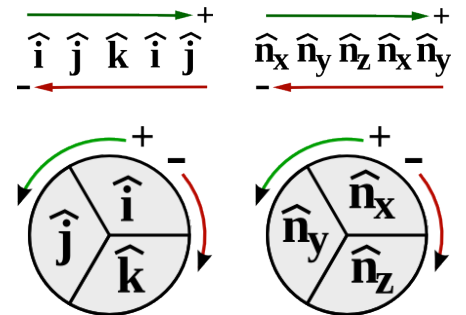
Graphical Visualization with Right-Hand Rule

A pictoral method is to **<u>draw</u>** a right-handed, orthonormal basis and then use the right-hand rule to deduce the cross-products between pairs of unit vectors. The figure below shows a few example sketches. In the two 3D views, note the coloring of the tips follows a common graphics convention where red, green, and blue (RGB) correspond to x-y-z (or i-j-k), respectively.
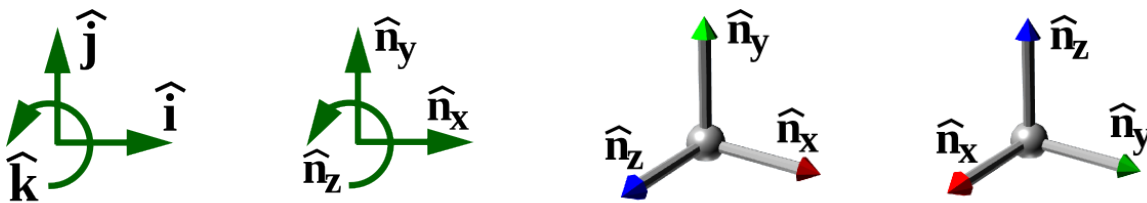
1. **Vector Basics - Dot and Cross Product via Distribution**

The figure to the right shows __unit__ vectors $\hat{\mathbf{v}}, \hat{\mathbf{w}}, \hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$.
The angle between adjacent vectors is 30°.

Given: $\quad \vec{\mathbf{a}} = 0.5\,\hat{\mathbf{y}} + 2\,\hat{\mathbf{w}} \qquad\qquad \vec{\mathbf{b}} = 2\,\hat{\mathbf{v}} - 1\,\hat{\mathbf{x}}$
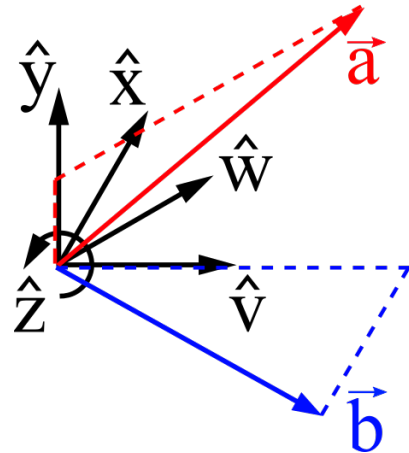
(a) __Draw__ vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ on the figure.

(b) Dot product can be regarded as a "***projection***" or "***measure***" of a vector in the direction of a unit vector.
   __Draw__ $\vec{\mathbf{a}} \cdot \hat{\mathbf{x}}$, the projection of $\vec{\mathbf{a}}$ on $\hat{\mathbf{x}}$ (the $\hat{\mathbf{x}}$ measure of $\vec{\mathbf{a}}$).

(c) Use the vector dot-product **distributive property** to calculate $\vec{\mathbf{a}} \cdot \hat{\mathbf{x}}$.
   Result:

$$\vec{\mathbf{a}} \cdot \hat{\mathbf{x}} = \boxed{2.5 * \cos(30°) = 2.17}$$

$$\vec{\mathbf{a}} \cdot \hat{\mathbf{x}} = (0.5\,\hat{\mathbf{y}} + 2\,\hat{\mathbf{w}}) \cdot \hat{\mathbf{x}}$$

$$= 0.5\,\hat{\mathbf{y}} \cdot \hat{\mathbf{x}} + 2\,\hat{\mathbf{w}} \cdot \hat{\mathbf{x}} \qquad\qquad \text{distribute}$$

$$= 0.5 * |\hat{\mathbf{y}}||\hat{\mathbf{x}}| \cos\angle(\hat{\mathbf{y}}, \hat{\mathbf{x}}) + 2\,|\hat{\mathbf{w}}||\hat{\mathbf{x}}| \cos\angle(\hat{\mathbf{w}}, \hat{\mathbf{x}}) \qquad \text{definition}$$

$$= 0.5 * (1)(1) \cos 30° + 2 * (1)(1) \cos 30° \qquad\qquad \text{plug in}$$

$$= 2.5 * \cos(30°) \qquad\qquad \text{simplify}$$

$$= 2.17$$
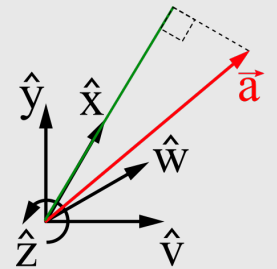
2.17 is the __length__ of the green line.

(d) Use the vector cross-product **distributive property** to calculate $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$.
   Result:

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \boxed{-3.75\,\hat{\mathbf{z}}}$$

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = (0.5\,\hat{\mathbf{y}} + 2\,\hat{\mathbf{w}}) \times (2\,\hat{\mathbf{v}} - 1\,\hat{\mathbf{x}})$$

$$= 1\,\hat{\mathbf{y}} \times \hat{\mathbf{v}} - 0.5\,\hat{\mathbf{y}} \times \hat{\mathbf{x}} + 4\,\hat{\mathbf{w}} \times \hat{\mathbf{v}} - 2\,\hat{\mathbf{w}} \times \hat{\mathbf{x}} \qquad\qquad \text{distribute}$$

$$= 1\,(-\hat{\mathbf{z}}) - 0.5\,(-\hat{\mathbf{z}}) \sin\angle(\hat{\mathbf{y}}, \hat{\mathbf{x}}) + 4\,(-\hat{\mathbf{z}}) \sin\angle(\hat{\mathbf{w}}, \hat{\mathbf{v}}) - 2\,(+\hat{\mathbf{z}}) \sin\angle(\hat{\mathbf{w}}, \hat{\mathbf{x}}) \qquad \text{definition}$$

$$= -1\,\hat{\mathbf{z}} + 0.5 \sin 30° \,\hat{\mathbf{z}} - 4\,\sin 30° \,\hat{\mathbf{z}} - 2\,\sin 30° \,\hat{\mathbf{z}} \qquad\qquad \text{plug in}$$

$$= -3.75\,\hat{\mathbf{z}} \qquad (\text{Note the result of a cross product must be a } \underline{\textbf{vector}}.) \qquad \text{simplify}$$

2. **Vector Basics - Measures of a Vector in Different Vector Bases**

(Adapted from problem 2.7.13 of Sheri Sheppard's textbook.)

The figure at right shows two cables tethered to a support at point O. The tension in the left cable is $T_1 = 2$ kip, and the tension in the right cable is $T_2 = 1$ kip (1 kip $=$ 1000 lb-force).

The support is embedded in concrete, which is very strong in compression but relatively weak in tension. Engineers are concerned that the support will pull out of the concrete if the force pulling perpendicular to the concrete surface (along the axis of the support, up-and-right) exceeds 2500 lb-force.
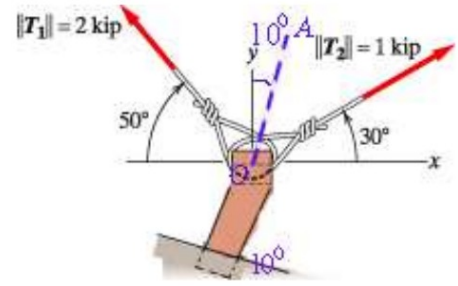
(a) Determine the resultant force at O due to the cables. Express your answer in terms of $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{y}}$ components, and then also in components parallel and perpendicular to the concrete surface. Note: Use the *definition* of dot-product.

> To aid in bookkeeping and to let us work with a *simple* expression, we define unit vectors in the direction of each cable: $\widehat{\mathbf{u}}$ in the direction of cable 1, and $\widehat{\mathbf{w}}$ in the direction of cable 2. The resultant force due to the cables is then:
>
> $$\vec{\mathbf{F}}^O = T_1\widehat{\mathbf{u}} + T_2\widehat{\mathbf{w}}$$
>
> This is the simplest possible expression of $\vec{\mathbf{F}}^O$. It is uncommitted to any particular components or measures.
>
> **Horizontal and Vertical components:**
>
> To express this vector in horizontal and vertical components, we can dot with $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{y}}$ to get the *measures* of the force in each of those directions. The angles in each dot-product are obtained by inspection of the figure.
>
> $$\vec{\mathbf{F}}^O \cdot \widehat{\mathbf{x}} = (T_1\widehat{\mathbf{u}} + T_2\widehat{\mathbf{w}}) \cdot \widehat{\mathbf{x}} = T_1\ \widehat{\mathbf{u}} \cdot \widehat{\mathbf{x}} + T_2\ \widehat{\mathbf{w}} \cdot \widehat{\mathbf{x}} = T_1\cos(130°) + T_2\cos(30°) = -0.420 \text{ kip}$$
>
> $$\vec{\mathbf{F}}^O \cdot \widehat{\mathbf{y}} = (T_1\widehat{\mathbf{u}} + T_2\widehat{\mathbf{w}}) \cdot \widehat{\mathbf{y}} = T_1\ \widehat{\mathbf{u}} \cdot \widehat{\mathbf{y}} + T_2\ \widehat{\mathbf{w}} \cdot \widehat{\mathbf{y}} = T_1\cos(40°) + T_2\cos(60°) = 2.03 \text{ kip}$$
>
> Hence $\vec{\mathbf{F}}^O$ can be written as:
>
> $$\vec{\mathbf{F}}^O = -0.420 \text{ kip } \widehat{\mathbf{x}} + 2.03 \text{ kip } \widehat{\mathbf{y}}$$
>
> **Components parallel and perpendicular to concrete surface:**
>
> To aid in bookkeeping, we define a new set of orthogonal unit vectors with $\widehat{\mathbf{c}}_x$ pointed parallel to the surface (down-and-right), and $\widehat{\mathbf{c}}_y$ pointed perpendicular to the surface (up-and-right).
>
> As before, to obtain the $\widehat{\mathbf{c}}_x$ and $\widehat{\mathbf{c}}_y$ *measures* of the force we simply dot with each of these unit vectors. The angles in each dot-product are obtained by inspection of the figure.
>
> $$\vec{\mathbf{F}}^O \cdot \widehat{\mathbf{c}}_x = (T_1\widehat{\mathbf{u}} + T_2\widehat{\mathbf{w}}) \cdot \widehat{\mathbf{c}}_x = T_1\ \widehat{\mathbf{u}} \cdot \widehat{\mathbf{c}}_x + T_2\ \widehat{\mathbf{w}} \cdot \widehat{\mathbf{c}}_x = T_1\cos(140°) + T_2\cos(40°) = -0.766 \text{ kip}$$
>
> $$\vec{\mathbf{F}}^O \cdot \widehat{\mathbf{c}}_y = (T_1\widehat{\mathbf{u}} + T_2\widehat{\mathbf{w}}) \cdot \widehat{\mathbf{c}}_y = T_1\ \widehat{\mathbf{u}} \cdot \widehat{\mathbf{c}}_y + T_2\ \widehat{\mathbf{w}} \cdot \widehat{\mathbf{c}}_y = T_1\cos(50°) + T_2\cos(50°) = 1.93 \text{ kip}$$
>
> Hence $\vec{\mathbf{F}}^O$ can be written as:
>
> $$\vec{\mathbf{F}}^O = -0.766 \text{ kip } \widehat{\mathbf{c}}_x + 1.93 \text{ kip } \widehat{\mathbf{c}}_y$$

(b) Use your results from the previous part to analyze the situation and determine if failure will occur.

> The engineers are concerned about the "force pulling along the axis of the support", which in our analysis is the $\widehat{\mathbf{c}}_y$ *measure* of $\vec{\mathbf{F}}^O$. We found the $\widehat{\mathbf{c}}_y$ measure to be 1.93 kip $=$ 1930 lb, so we know the concrete will not fail.
>
> **Note:** It is **not necessary** to fully resolve a vector into components in order to determine one of them. If all we really care about is the $\widehat{\mathbf{c}}_y$ measure of the force, then we simply dot with $\widehat{\mathbf{c}}_y$:
>
> "force along axis of support" $= \vec{\mathbf{F}}^O \cdot \widehat{\mathbf{c}}_y = T_1\ \widehat{\mathbf{u}} \cdot \widehat{\mathbf{c}}_y + T_2\ \widehat{\mathbf{w}} \cdot \widehat{\mathbf{c}}_y = T_1\cos(50°) + T_2\cos(50°) = 1.93 \text{ kip}$

3. **Vector Basics - Cross Product to Calculate Moments**

(Adapted from problem 3.2.38 of Sheri Sheppard's textbook.)

The figure at right shows a cable connected to a lever. The tension in the cable is $T = 610\,\text{N}$. Determine the moment about point O created by the force of the cable acting on the lever. Express your answer in terms of unit vectors $\widehat{\mathbf{x}}$, $\widehat{\mathbf{y}}$, and $\widehat{\mathbf{z}}$ which are directed along the axes shown.
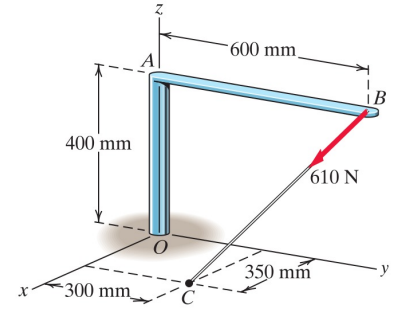
We'll denote the force applied at point B as $\vec{\mathbf{F}}^B$. Using the *definition*, the moment of this force about point O is:

$$\vec{\mathbf{M}}^{\vec{\mathbf{F}}^B/O} \triangleq {}^O\vec{\mathbf{r}}^B \times \vec{\mathbf{F}}^B$$

Position vector: The position vector is formed by direct inspection of the figure. If this is not obvious to you, place your finger at O and trace a path to B, writing down the distance and direction you moved at each step.

$$ {}^O\vec{\mathbf{r}}^B = {}^O\vec{\mathbf{r}}^A + {}^A\vec{\mathbf{r}}^B = 400\text{ mm }\widehat{\mathbf{z}} + 600\text{ mm }\widehat{\mathbf{y}}$$

Force vector: We are given the tension in the cable, so we know the magnitude of the force is $610\,\text{N}$. We do not know the direction, but we have geometric information in the figure we can use to find the direction. The line of action of the force passes through points B and C, so to form the force we need the unit vector from B to C, denoted $\widehat{\mathbf{u}}^{C/B}$.

$$\vec{\mathbf{F}}^B = T\,\widehat{\mathbf{u}}^{C/B} = T\frac{{}^B\vec{\mathbf{r}}^C}{\left|{}^B\vec{\mathbf{r}}^C\right|}$$

As before, we find ${}^B\vec{\mathbf{r}}^C$ by inspection:

$$ {}^B\vec{\mathbf{r}}^C = {}^O\vec{\mathbf{r}}^C + {}^B\vec{\mathbf{r}}^O = (350\,\widehat{\mathbf{x}} + 300\,\widehat{\mathbf{y}}) + (-600\,\widehat{\mathbf{y}} + -400\,\widehat{\mathbf{z}}) = 350\text{ mm }\widehat{\mathbf{x}} + -300\text{ mm }\widehat{\mathbf{y}} + -400\text{ mm }\widehat{\mathbf{z}}$$

And we find the magnitude from the definition:

$$\left|{}^B\vec{\mathbf{r}}^C\right| \triangleq \sqrt{{}^B\vec{\mathbf{r}}^C \cdot {}^B\vec{\mathbf{r}}^C} = \underbrace{\sqrt{(350\text{ mm})^2 + (-300\text{ mm})^2 + (-400\text{ mm})^2}}_{\text{(Valid for vector expressed in a single, unitary orthogonal basis.)}} = 610.3\text{ mm}$$

Hence, the force vector is

$$\vec{\mathbf{F}}^B = T\,\widehat{\mathbf{u}}^{C/B} = (610\text{ N})\frac{(350\text{ mm }\widehat{\mathbf{x}} + -300\text{ mm }\widehat{\mathbf{y}} + -400\text{ mm }\widehat{\mathbf{z}})}{610.3\text{ mm}} \approx 350\text{ N }\widehat{\mathbf{x}} + -300\text{ N }\widehat{\mathbf{y}} + -400\text{ N }\widehat{\mathbf{z}}.$$

Finally, we have the ingredients to compute the moment:

$$\vec{\mathbf{M}}^{\vec{\mathbf{F}}^B/O} \triangleq {}^O\vec{\mathbf{r}}^B \times \vec{\mathbf{F}}^B$$

$$= (600\text{ mm }\widehat{\mathbf{y}} + 400\text{ mm }\widehat{\mathbf{z}}) \times (350\text{ N }\widehat{\mathbf{x}} + -300\text{ N }\widehat{\mathbf{y}} + -400\text{ N }\widehat{\mathbf{z}})$$

$$= (\,(600)(350)\underbrace{\widehat{\mathbf{y}}\times\widehat{\mathbf{x}}}_{-\widehat{\mathbf{z}}} + (600)(-300)\underbrace{\widehat{\mathbf{y}}\times\widehat{\mathbf{y}}}_{\vec{0}} + (600)(-400)\underbrace{\widehat{\mathbf{y}}\times\widehat{\mathbf{z}}}_{+\widehat{\mathbf{x}}} +$$

$$(400)(350)\underbrace{\widehat{\mathbf{z}}\times\widehat{\mathbf{x}}}_{+\widehat{\mathbf{y}}} + (400)(-300)\underbrace{\widehat{\mathbf{z}}\times\widehat{\mathbf{y}}}_{-\widehat{\mathbf{x}}} + (400)(-400)\underbrace{\widehat{\mathbf{z}}\times\widehat{\mathbf{z}}}_{\vec{0}}\,)\text{ N mm}$$

$$= (-120,000\,\widehat{\mathbf{x}} + 140,000\,\widehat{\mathbf{y}} + -210,000\,\widehat{\mathbf{z}})\text{ N mm}$$

$$= (-120\,\widehat{\mathbf{x}} + 140\,\widehat{\mathbf{y}} + -210\,\widehat{\mathbf{z}})\text{ N m}$$

Addendum: A force is applied at a specific *physical* point of a body, which is why moments are defined using a position vector to the point of application. However, *mathematically* we can use any point that is on the force's "line of action". In some problems this can dramatically simplify the math.

In this problem the math isn't any simpler, but we can show that picking the point on the ground gives us the same result as before (note the use of the equal sign; this is <u>not</u> a definition):

$$
\begin{aligned}
\vec{\mathbf{M}}^{\vec{\mathbf{F}}^B/O} &= {}^{O}\vec{\mathbf{r}}^{C} \times \vec{\mathbf{F}}^{B} \\
&= (350 \text{ mm } \widehat{\mathbf{x}} + 300 \text{ mm } \widehat{\mathbf{y}}) \times (350 \text{ N } \widehat{\mathbf{x}} + -300 \text{ N } \widehat{\mathbf{y}} + -400 \text{ N } \widehat{\mathbf{z}}) \\
&= (\ (350)(350)\,\widehat{\mathbf{x}}\times\widehat{\mathbf{x}}^{\ \vec{0}} + (350)(-300)\,\widehat{\mathbf{x}}\times\widehat{\mathbf{y}}^{\ +\widehat{\mathbf{z}}} + (350)(-400)\,\widehat{\mathbf{x}}\times\widehat{\mathbf{z}}^{\ -\widehat{\mathbf{y}}} + \\
&\qquad (300)(350)\,\widehat{\mathbf{y}}\times\widehat{\mathbf{x}}^{\ -\widehat{\mathbf{z}}} + (300)(-300)\,\widehat{\mathbf{y}}\times\widehat{\mathbf{y}}^{\ \vec{0}} + (300)(-400)\,\widehat{\mathbf{y}}\times\widehat{\mathbf{z}}^{\ +\widehat{\mathbf{x}}}\ )\text{ N mm} \\
&= (-120{,}000\ \widehat{\mathbf{x}} + 140{,}000\ \widehat{\mathbf{y}} + -210{,}000\ \widehat{\mathbf{z}})\text{ N mm} \\
&= (-120\ \widehat{\mathbf{x}} + 140\ \widehat{\mathbf{y}} + -210\ \widehat{\mathbf{z}})\text{ N m}
\end{aligned}
$$

4. **Vector Equations - Solving for Unknowns (Statics)**

Consider the block on a *smooth* inclined plane as shown at right. Let system S be the block.

<u>**Draw**</u> a FBD of S. State your assumptions.

- Only consider gravity from earth (down).
- Spring is linear.
- Block is modeled as a particle.
- Smooth plane has negligible friction.
- Air, thermal, and other effects negligible.
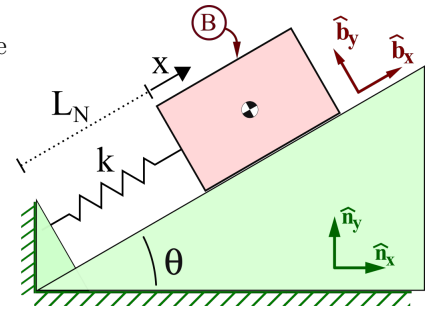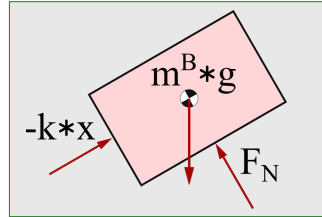
Using $\vec{\mathbf{F}}^S = \vec{\mathbf{0}}$ might lead to the following vector equation (based on assumptions about relevant forces):

$$-kx\,\widehat{\mathbf{b}}_{\mathrm{x}} \;+\; F_N\,\widehat{\mathbf{b}}_{\mathrm{y}} \;-\; mg\,\widehat{\mathbf{n}}_{\mathrm{y}} \;=\; \vec{\mathbf{0}}$$

Note: It is advantageous to express vectors in the <u>**simplest**</u> terms possible. Hence, two bases were introduced, one aligned with the inclined plane, and one aligned with gravity, to avoid having trig functions appear in the vector equation. Avoiding trig functions improves efficiency and helps reduce errors.

Given the equation above:

(a) Find a scalar expression for F_N (the magnitude of the "normal force" on the block from the plane).

> **The unknown we want is in a $\widehat{\mathbf{b}}_{\mathrm{y}}$ term, so we'll dot both sides with $\widehat{\mathbf{b}}_{\mathrm{y}}$.**
>
> $$\widehat{\mathbf{b}}_{\mathrm{y}} \cdot (-kx\,\widehat{\mathbf{b}}_{\mathrm{x}} \;+\; F_N\,\widehat{\mathbf{b}}_{\mathrm{y}} \;-\; mg\,\widehat{\mathbf{n}}_{\mathrm{y}}) \;=\; \widehat{\mathbf{b}}_{\mathrm{y}} \cdot \vec{\mathbf{0}}$$
> $$-kx\,\widehat{\mathbf{b}}_{\mathrm{y}} \cdot \widehat{\mathbf{b}}_{\mathrm{x}} \;+\; F_N\,\widehat{\mathbf{b}}_{\mathrm{y}} \cdot \widehat{\mathbf{b}}_{\mathrm{y}} \;-\; mg\,\widehat{\mathbf{b}}_{\mathrm{y}} \cdot \widehat{\mathbf{n}}_{\mathrm{y}} \;=\; \widehat{\mathbf{b}}_{\mathrm{y}} \cdot \vec{\mathbf{0}} \qquad \leftarrow \text{Find dot products from definition.}$$
> $$0 \;+\; F_N \;-\; mg\cos(\theta) \;=\; 0$$
> $$F_N \;=\; mg\cos(\theta)$$

(b) Find a scalar expression for x. (This gives the equilibrium stretch or compression of the spring).

> **The unknown we want is in a $\widehat{\mathbf{b}}_{\mathrm{x}}$ term, so we'll dot both sides with $\widehat{\mathbf{b}}_{\mathrm{x}}$.**
>
> $$\widehat{\mathbf{b}}_{\mathrm{x}} \cdot (-kx\,\widehat{\mathbf{b}}_{\mathrm{x}} \;+\; F_N\,\widehat{\mathbf{b}}_{\mathrm{y}} \;-\; mg\,\widehat{\mathbf{n}}_{\mathrm{y}}) \;=\; \widehat{\mathbf{b}}_{\mathrm{x}} \cdot \vec{\mathbf{0}}$$
> $$-kx\,\widehat{\mathbf{b}}_{\mathrm{x}} \cdot \widehat{\mathbf{b}}_{\mathrm{x}} \;+\; F_N\,\widehat{\mathbf{b}}_{\mathrm{x}} \cdot \widehat{\mathbf{b}}_{\mathrm{y}} \;-\; mg\,\widehat{\mathbf{b}}_{\mathrm{x}} \cdot \widehat{\mathbf{n}}_{\mathrm{y}} \;=\; \widehat{\mathbf{b}}_{\mathrm{x}} \cdot \vec{\mathbf{0}} \qquad \leftarrow \text{Find dot products from definition.}$$
> $$-kx \;+\; 0 \;-\; mg\sin(\theta) \;=\; 0$$
> $$x \;=\; -\tfrac{m}{k}g\sin(\theta)$$

Note 1: In general, it is <u>**not correct**</u> to "equate components" to create scalar equations from a vector equation. In this example, doing so would lead to the (totally wrong) equations $-kx = 0$, $F_N = 0$, and $-mg = 0$. This idea is taught in calculus because it works when the directions in the equation are mutually orthogonal, but it clearly breaks when there are non-orthogonal vectors and/or multiple bases.

Note 2: In this example, the gravity force was <u>**not**</u> resolved into $\widehat{\mathbf{b}}_{\mathrm{x}}$ and $\widehat{\mathbf{b}}_{\mathrm{y}}$ components in the vector equation. Instead, the familiar $mg\cos(\theta)$ and $mg\sin(\theta)$ terms came from dotting. This is a <u>**key insight**</u> to be able to do complex and/or 3D problems efficiently. Resolving into components using trig may be nearly impossible at times, but looking up individual dot-products in one or more rotation tables is easy.

Chapter 2

Rotation Tables

1. **Vector Geometry - Position Measures**

   Consider the robot arm shown. Frames A, B, C (shown) and frame N (not shown) each have fixed in them right-handed, orthogonal unit vectors $\widehat{\mathbf{a}}_{xyz}$, $\widehat{\mathbf{b}}_{xyz}$, $\widehat{\mathbf{c}}_{xyz}$, and $\widehat{\mathbf{n}}_{xyz}$, respectively. The relevant lengths L_1, L_2, and L_3 are shown on the drawing.

   You are **given** the vector from a point N_o (not shown) on the robot's base to the elbow point A_o is $^{N_o}\vec{\mathbf{r}}^{A_o} = d_y\,\widehat{\mathbf{n}}_y + d_z\,\widehat{\mathbf{n}}_z$.

   (a) By inspection, write the position vector **from** the elbow point A_o **to** the finger tip Q, using an efficient combination of basis vectors.

   $$^{A_o}\vec{\mathbf{r}}^Q = \boxed{L_1\,\widehat{\mathbf{a}}_x + L_2\,\widehat{\mathbf{b}}_x + L_3\,\widehat{\mathbf{c}}_y}$$

   (b) Write the position vector from N_o to Q, first as a sum of two or more **basis-independent** position vectors, then in terms of any given unit vectors.

   $$^{N_o}\vec{\mathbf{r}}^Q = \boxed{^{A_o}\vec{\mathbf{r}}^Q} + \boxed{^{N_o}\vec{\mathbf{r}}^{A_o}}$$

   $$^{N_o}\vec{\mathbf{r}}^Q = \boxed{d_y\,\widehat{\mathbf{n}}_y + d_z\,\widehat{\mathbf{n}}_z + L_1\,\widehat{\mathbf{a}}_x + L_2\,\widehat{\mathbf{b}}_x + L_3\,\widehat{\mathbf{c}}_y}$$
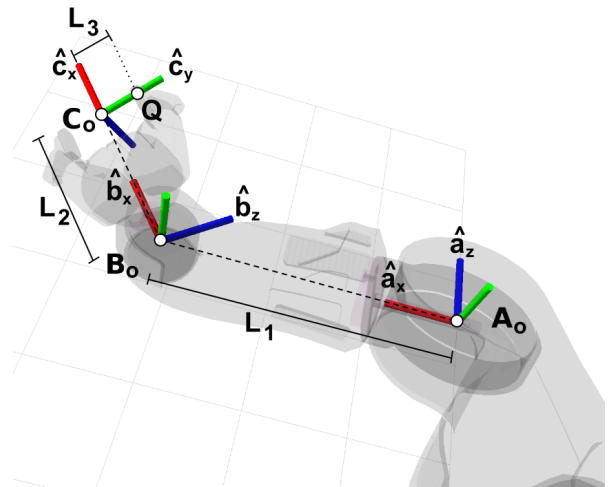
   (c) Your team-mate asks you for the "x, y, z coordinates" of Q's position from N_o in frame N. Explain/show which vector operations you would do to get these **measures** of $^{N_o}\vec{\mathbf{r}}^Q$. (You do not need to complete the calculation; in fact, you don't have enough information to do so.)

   > $\widehat{\mathbf{n}}_x$ measure $= (\widehat{\mathbf{n}}_x \cdot {}^{N_o}\vec{\mathbf{r}}^Q)$
   > $\widehat{\mathbf{n}}_y$ measure $= (\widehat{\mathbf{n}}_y \cdot {}^{N_o}\vec{\mathbf{r}}^Q)$
   > $\widehat{\mathbf{n}}_z$ measure $= (\widehat{\mathbf{n}}_z \cdot {}^{N_o}\vec{\mathbf{r}}^Q)$
   >
   > **Note:** Can't complete the dot products, need rotation matrices relating the frames.

   (d) A heavy ball held at Q exerts a downward force $\vec{\mathbf{F}}^Q = -W\,\widehat{\mathbf{n}}_z$ on the robot arm. Explain/show which (one) vector operation you would do to get the moment of this force about A_o, that is, $\vec{\mathbf{M}}^{F^Q/A_o}$.

   > $\vec{\mathbf{M}}^{F^Q/A_o} \triangleq {}^{A_o}\vec{\mathbf{r}}^Q \times \vec{\mathbf{F}}^Q = (L_1\,\widehat{\mathbf{a}}_x + L_2\,\widehat{\mathbf{b}}_x + L_3\,\widehat{\mathbf{c}}_y) \times (-W\,\widehat{\mathbf{n}}_z)$
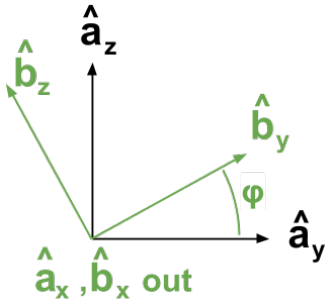   >
   > **Note:** Could distribute, but can't complete the cross products. Need rotation matrices relating the frames.

2. **Rotation Tables - Simple Rotation**

Frames A and B, with **right-handed** orthogonal unit vectors $\widehat{\mathbf{a}}_{xyz}$, and $\widehat{\mathbf{b}}_{xyz}$, respectively, are initially aligned. Frame B is then rotated by an angle ϕ about the positive $\widehat{\mathbf{a}}_x$ axis.

(a) Sketch $\widehat{\mathbf{a}}_{xyz}$, $\widehat{\mathbf{b}}_{xyz}$, and ϕ on the figure below.

(b) Write a rotation table relating A and B.

$^AR^B$	$\widehat{\mathbf{b}}_x$	$\widehat{\mathbf{b}}_y$	$\widehat{\mathbf{b}}_z$
$\widehat{\mathbf{a}}_x$	1	0	0
$\widehat{\mathbf{a}}_y$	0	$\cos(\phi)$	$-\sin(\phi)$
$\widehat{\mathbf{a}}_z$	0	$\sin(\phi)$	$\cos(\phi)$

3. **Rotation Tables - Matrix Manipulation**

In computer graphics and robotics, a *scene graph* is a structure which holds position and rotation information for a set of frames. Typically this structure gives you rotation matrices relating each frame to a single common "world" frame; it is then up to you to compute the quantities you need for a particular use.

For this problem, assume you obtain from the scene graph the rotation matrices $^NR^A$, $^NR^B$, $^NR^C$ which relate the right-handed, orthogonal unit vectors of frames N, A, B, and C.

(a) Given $^NR^A$, explain how to obtain the inverse rotation matrix, $^AR^N$.

> You <u>transpose</u> $^NR^A$ to get $^AR^N$. In other words, $^AR^N = (^NR^A)^T$.
>
> Note: In linear algebra, a matrix with the property $^NR^A * {}^AR^N = I$ is called an *orthogonal* matrix.

(b) Show how to use the above quantities to form the rotation matrix relating frame A and C.

$$^AR^C = \boxed{{}^AR^N * {}^NR^C = (^NR^A)^T * {}^NR^C}$$

4. **Rotation Tables - Frame Conventions**

Creating a flight simulator requires cooperation of engineers from multiple disciplines (graphics and aeronautics) that have different frame conventions. Many 3D graphics applications consider the "world" frame to be defined as shown by vector basis C, where the "ground" plane is defined by $\widehat{\mathbf{c}}_x$ and $\widehat{\mathbf{c}}_z$, with $\widehat{\mathbf{c}}_y$ pointing opposite gravity. Meanwhile, aeronautics uses a frame convention shown by vector basis D, where $\widehat{\mathbf{d}}_x$ points north, $\widehat{\mathbf{d}}_y$ points east, and $\widehat{\mathbf{d}}_z$ points in the direction of gravity (this convention is often called NED, for "north-east-down"). For clarity, in the picture below the frames C and D are "aligned" but use a different convention. Note $\widehat{\mathbf{c}}_z$ is "out" of the page and $\widehat{\mathbf{d}}_x$ is "in."

To properly display aeronautics simulation data (computed and expressed in D) in the graphics engine, the data must be re-expressed using the basis vectors of frame C. Form the rotation table $^CR^d$ relating $\widehat{\mathbf{c}}_x$, $\widehat{\mathbf{c}}_y$, $\widehat{\mathbf{c}}_z$ to $\widehat{\mathbf{d}}_x$, $\widehat{\mathbf{d}}_y$, $\widehat{\mathbf{d}}_z$.
Note: The elements of the rotation table are numbers (no symbols).
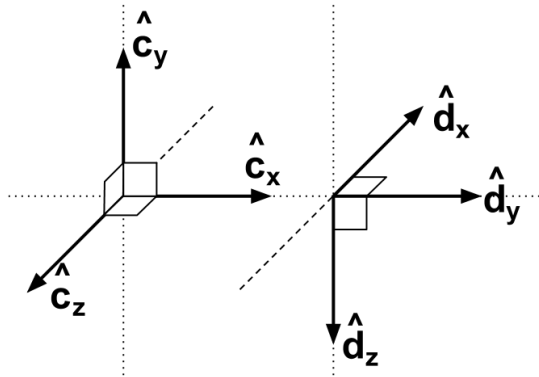
$^CR^D$	$\widehat{\mathbf{d}}_x$	$\widehat{\mathbf{d}}_y$	$\widehat{\mathbf{d}}_z$
$\widehat{\mathbf{c}}_x$	0	1	0
$\widehat{\mathbf{c}}_y$	0	0	−1
$\widehat{\mathbf{c}}_z$	−1	0	0

5. Rotation Tables - Mixed-Basis Vector Operations

Let $\widehat{\mathbf{n}}_{xyz}$ and $\widehat{\mathbf{b}}_{xyz}$ be two distinct sets of right-handed, orthogonal unit vectors related by the rotation table ${}^{\mathrm{n}}R^{\mathrm{b}}$ given at right. The entries c_{ij} in the matrix represent scalars.

Assume you are **given** a vector ${}^P\vec{\mathbf{r}}^{\,Q} = 3\,\widehat{\mathbf{n}}_x + 2\,\widehat{\mathbf{b}}_z$.

${}^{\mathrm{n}}R^{\mathrm{b}}$	$\widehat{\mathbf{b}}_x$	$\widehat{\mathbf{b}}_y$	$\widehat{\mathbf{b}}_z$
$\widehat{\mathbf{n}}_x$	c_{xx}	c_{xy}	c_{xz}
$\widehat{\mathbf{n}}_y$	c_{yx}	c_{yy}	c_{yz}
$\widehat{\mathbf{n}}_z$	c_{zx}	c_{zy}	c_{zz}

(a) Calculate ${}^P\vec{\mathbf{r}}^{\,Q} \cdot {}^P\vec{\mathbf{r}}^{\,Q}$, expressing your answer in terms of the coefficients c_{ij}.

$$
\begin{aligned}
{}^P\vec{\mathbf{r}}^{\,Q} \cdot {}^P\vec{\mathbf{r}}^{\,Q} &= (3\,\widehat{\mathbf{n}}_x + 2\,\widehat{\mathbf{b}}_z) \cdot (3\,\widehat{\mathbf{n}}_x + 2\,\widehat{\mathbf{b}}_z) \\
&= 9\,\widehat{\mathbf{n}}_x \cdot \widehat{\mathbf{n}}_x + 6\,\widehat{\mathbf{n}}_x \cdot \widehat{\mathbf{b}}_z + 6\,\widehat{\mathbf{b}}_z \cdot \widehat{\mathbf{n}}_x + 4\,\widehat{\mathbf{b}}_z \cdot \widehat{\mathbf{b}}_z \quad &\text{distribute, preserving order} \\
&= 9\,\widehat{\mathbf{n}}_x \cdot \widehat{\mathbf{n}}_x + 12\,\widehat{\mathbf{n}}_x \cdot \widehat{\mathbf{b}}_z + 4\,\widehat{\mathbf{b}}_z \cdot \widehat{\mathbf{b}}_z \quad &\text{dot product is commutative} \\
&= 9\,(1) + 12\,(c_{13}) + 4\,(1) \quad &\text{look up dot products} \\
&= 13 + 12\,c_{13} \quad &\text{simplify}
\end{aligned}
$$

(b) Calculate $\widehat{\mathbf{n}}_z \times {}^P\vec{\mathbf{r}}^{\,Q}$, expressing your answer in terms of $\widehat{\mathbf{n}}_{xyz}$ and the coefficients c_{ij}.

$$
\begin{aligned}
\widehat{\mathbf{n}}_z \times {}^P\vec{\mathbf{r}}^{\,Q} &= \widehat{\mathbf{n}}_z \times (3\,\widehat{\mathbf{n}}_x + 2\,\widehat{\mathbf{b}}_z) \\
&= 3\,\widehat{\mathbf{n}}_z \times \widehat{\mathbf{n}}_x + 2\,\widehat{\mathbf{n}}_z \times \widehat{\mathbf{b}}_z \quad &\text{distribute, preserve order} \\
&= 3\,\widehat{\mathbf{n}}_z \times \widehat{\mathbf{n}}_x + 2\,\widehat{\mathbf{n}}_z \times (c_{13}\,\widehat{\mathbf{n}}_x + c_{23}\,\widehat{\mathbf{n}}_y + c_{33}\,\widehat{\mathbf{n}}_z) \quad &\text{convert } \widehat{\mathbf{b}}_z \text{ with table} \\
&= 3\,\widehat{\mathbf{n}}_z \times \widehat{\mathbf{n}}_x + 2\,c_{13}\,\widehat{\mathbf{n}}_z \times \widehat{\mathbf{n}}_x + 2\,c_{23}\,\widehat{\mathbf{n}}_z \times \widehat{\mathbf{n}}_y + 2\,c_{33}\,\widehat{\mathbf{n}}_z \times \widehat{\mathbf{n}}_z \quad &\text{distribute, preserve order} \\
&= 3\,\widehat{\mathbf{n}}_y + 2\,c_{13}\,\widehat{\mathbf{n}}_y - 2\,c_{23}\,\widehat{\mathbf{n}}_x + \vec{\mathbf{0}} \quad &\text{evaluate cross products} \\
&= -2\,c_{23}\,\widehat{\mathbf{n}}_x + (3 + 2\,c_{13})\,\widehat{\mathbf{n}}_y \quad &\text{simplify}
\end{aligned}
$$

Chapter 3

Velocity and Acceleration

3.1 Example: Deriving Kinematics Relationships for "Circular" Motion

In this section we show clearly how to derive velocity and acceleration for a particle whose position is described by polar coordinates r and θ. In many "traditional" textbooks (Hibbeler, Beer and Johnson, *etc.*) the mechanics of vector derivatives with rotating frames are not taught, so the kinematics expressions are simply given. However, deriving these quantities is a straight-forward application of the golden rule.

To ground the development with a concrete example, consider the figure shown at right. The disk is modeled as a rigid body B, and its center B_o is fixed in a Newtonian frame N and co-incident with N_o. We define our usual right-handed, orthogonal unit vectors $\widehat{\mathbf{n}}_{\mathrm{x}}$, $\widehat{\mathbf{n}}_{\mathrm{y}}$, $\widehat{\mathbf{n}}_{\mathrm{z}}$ fixed in N with $\widehat{\mathbf{n}}_{\mathrm{x}}$ horizontally right and $\widehat{\mathbf{n}}_{\mathrm{y}}$ vertically upward. Similarly, $\widehat{\mathbf{b}}_{\mathrm{x}}$, $\widehat{\mathbf{b}}_{\mathrm{y}}$, $\widehat{\mathbf{b}}_{\mathrm{z}}$ are fixed in B with $\widehat{\mathbf{b}}_{\mathrm{z}} = \widehat{\mathbf{n}}_{\mathrm{z}}$ and $\widehat{\mathbf{b}}_{\mathrm{x}}$ pointing from B_o to Q.
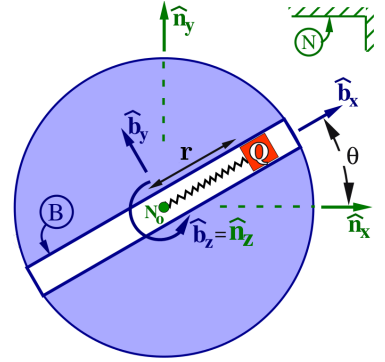
3.1.1 General Polar Kinematics

We will differentiate the position vector to find velocity and acceleration (note the "two-points-fixed" formulas don't apply because Q is not fixed on B). We will need the angular velocity and angular acceleration of B in N, which is formed **by inspection** since B has a **simple** rotation in N about $\widehat{\mathbf{b}}_{\mathrm{z}}$: $^{N}\vec{\boldsymbol{\omega}}^{B} = \dot{\theta}\,\widehat{\mathbf{b}}_{\mathrm{z}}$ and $^{N}\vec{\boldsymbol{\alpha}}^{B} = \ddot{\theta}\,\widehat{\mathbf{b}}_{\mathrm{z}}$.

Acceleration via Differentiation of Position Vector

$$^{O}\vec{\mathbf{r}}^{P} = r\,\widehat{\mathbf{b}}_{\mathrm{x}}$$

$$
\begin{aligned}
^{N}\vec{\mathbf{v}}^{P} &\triangleq \frac{^{N}d}{dt}\left(\,^{O}\vec{\mathbf{r}}^{P}\right) && \text{This is the } \textbf{definition}. \\
&= \frac{^{N}d}{dt}\left(\,r\,\widehat{\mathbf{b}}_{\mathrm{x}}\right) && \text{So we } \textbf{need} \text{ to differentiate this in } N. \\
&= \frac{^{B}d}{dt}\left(\,r\,\widehat{\mathbf{b}}_{\mathrm{x}}\right) + \,^{N}\vec{\boldsymbol{\omega}}^{B} \times \left(\,r\,\widehat{\mathbf{b}}_{\mathrm{x}}\right) && \text{It's } \textbf{easier} \text{ to differentiate this in } B, \text{ so we use golden rule to convert.} \\
&= \dot{r}\,\widehat{\mathbf{b}}_{\mathrm{x}} + (\dot{\theta}\,\widehat{\mathbf{b}}_{\mathrm{z}}) \times \left(\,r\,\widehat{\mathbf{b}}_{\mathrm{x}}\right) && \widehat{\mathbf{b}}_{\mathrm{x}} \text{ is fixed in } B, \text{ so we can just differentiate the scalar portion.} \\
&= \dot{r}\,\widehat{\mathbf{b}}_{\mathrm{x}} + r\dot{\theta}\,\widehat{\mathbf{b}}_{\mathrm{y}} && \text{And finally we do the cross product.}
\end{aligned}
$$

$$
\begin{aligned}
^{N}\vec{\mathbf{a}}^{P} &\triangleq \frac{^{N}d}{dt}\left(\,^{N}\vec{\mathbf{v}}^{P}\right) && \text{This is the } \textbf{definition}. \\
&= \frac{^{N}d}{dt}\left(\,\dot{r}\,\widehat{\mathbf{b}}_{\mathrm{x}} + r\dot{\theta}\,\widehat{\mathbf{b}}_{\mathrm{y}}\right) && \text{So we } \textbf{need} \text{ to differentiate this in } N. \\
&= \frac{^{B}d}{dt}\left(\,\dot{r}\,\widehat{\mathbf{b}}_{\mathrm{x}} + r\dot{\theta}\,\widehat{\mathbf{b}}_{\mathrm{y}}\right) + \,^{N}\vec{\boldsymbol{\omega}}^{B} \times \left(\,\dot{r}\,\widehat{\mathbf{b}}_{\mathrm{x}} + r\dot{\theta}\,\widehat{\mathbf{b}}_{\mathrm{y}}\right) && \textbf{Easier} \text{ to differentiate in } B, \text{ so apply golden rule.} \\
&= \left(\,\ddot{r}\,\widehat{\mathbf{b}}_{\mathrm{x}} + (\dot{r}\dot{\theta} + r\ddot{\theta})\,\widehat{\mathbf{b}}_{\mathrm{y}}\right) + (\dot{\theta}\,\widehat{\mathbf{b}}_{\mathrm{z}}) \times \left(\,\dot{r}\,\widehat{\mathbf{b}}_{\mathrm{x}} + r\dot{\theta}\,\widehat{\mathbf{b}}_{\mathrm{y}}\right) && \text{Differentiate scalars (remember } \textbf{product rule}!\text{).} \\
&= \left(\,\ddot{r}\,\widehat{\mathbf{b}}_{\mathrm{x}} + (\dot{r}\dot{\theta} + r\ddot{\theta})\,\widehat{\mathbf{b}}_{\mathrm{y}}\right) + \left(\dot{r}\dot{\theta}\,(\widehat{\mathbf{b}}_{\mathrm{y}}) + r\dot{\theta}^2\,(-\widehat{\mathbf{b}}_{\mathrm{x}})\right) && \text{Do cross product.} \\
&= \left(\,\ddot{r} - r\dot{\theta}^2\right)\widehat{\mathbf{b}}_{\mathrm{x}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\widehat{\mathbf{b}}_{\mathrm{y}} && \text{Collect terms.}
\end{aligned}
$$

Now that we have general expressions velocity and acceleration of a particle in planar motion expressed in polar coordinates, let's see how this can be specialized for some common scenarios.

3.1.2 Curvilinear Motion: Polar/Cylindrical Components

Paraphrasing loosely, a "traditional" book may say something like:

> "For problems described by polar coordinates, you **must** assign a coordinate system with $\widehat{\mathbf{e}}_r$ radially outward, $\widehat{\mathbf{e}}_\theta$ perpendicular to $\widehat{\mathbf{e}}_r$ and in the direction of increasing θ, and $\widehat{\mathbf{e}}_z = \widehat{\mathbf{e}}_r \times \widehat{\mathbf{e}}_\theta$." (In our example, $\widehat{\mathbf{e}}_r = \widehat{\mathbf{b}}_{\mathrm{x}}$, $\widehat{\mathbf{e}}_\theta = \widehat{\mathbf{b}}_{\mathrm{y}}$, and $\widehat{\mathbf{e}}_z = \widehat{\mathbf{b}}_{\mathrm{z}}$.)

Then you would be given the following scalar values for the measures/components of acceleration:

$$a_r = \ddot{r} - r\dot{\theta}^2$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

How do we get there from our expressions for $^{N}\vec{\mathbf{v}}^{Q}$ and $^{N}\vec{\mathbf{a}}^{Q}$? In this class we get measures of a vector by dotting. We can dot $^{N}\vec{\mathbf{a}}^{Q}$ with $(\widehat{\mathbf{e}}_r = \widehat{\mathbf{b}}_{\mathrm{x}})$ and $(\widehat{\mathbf{e}}_\theta = \widehat{\mathbf{b}}_{\mathrm{y}})$ to get these same expressions:

$$a_r = {}^{N}\vec{\mathbf{a}}^Q \cdot \widehat{\mathbf{e}}_r = \ddot{r} - r\dot{\theta}^2$$

$$a_\theta = {}^{N}\vec{\mathbf{a}}^Q \cdot \widehat{\mathbf{e}}_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

It is **ok** to use these pre-computed expressions if you are **sure** that you have the signs correct. However, it is often convenient to assign frames in a different way (for example, in problem 7.10 frame B has $\widehat{\mathbf{e}}_r = \widehat{\mathbf{b}}_y$ and $\widehat{\mathbf{e}}_\theta = \widehat{\mathbf{b}}_x$ and in problem 7.9 and 7.16 frame B has $\widehat{\mathbf{e}}_r = -\widehat{\mathbf{b}}_y$ and $\widehat{\mathbf{e}}_\theta = \widehat{\mathbf{b}}_x$) so you cannot blindly use these expressions without accounting for this.

3.1.3 Curvilinear Motion: Normal-Tangential Components

Paraphrasing loosely, a "traditional" book may say something like:

> "When the path of a particle is **known**, it is possible to approximate its motion during a small interval as motion along a circular arc with radius r. You **must** assign a coordinate system with $\widehat{\mathbf{u}}_n$ radially **inward** and $\widehat{\mathbf{u}}_t$ tangent to the circular path in the direction of the particle's velocity." (In our example, $\widehat{\mathbf{u}}_n = -\widehat{\mathbf{b}}_x$ and $\widehat{\mathbf{u}}_t = \widehat{\mathbf{b}}_y$.)

Two implicit requirements of this model are that **during the short interval being considered**:

- the radius of the path is **constant** ($\dot{r} = \ddot{r} = 0$).

- the particle's velocity is purely tangential, described by a scalar v, where $v \triangleq {}^{N}\vec{\mathbf{v}}^Q \cdot \widehat{\mathbf{u}}_t$.

Then you would be given the following scalar values for the measures/components of acceleration:

$$a_n = v^2/r$$

$$a_t = \dot{v}$$

How do we get there from our expressions for ${}^{N}\vec{\mathbf{v}}^Q$ and ${}^{N}\vec{\mathbf{a}}^Q$? If we plug in $\dot{r} = \ddot{r} = 0$ we get:

$$^{N}\vec{\mathbf{v}}^Q = (0)\,\widehat{\mathbf{b}}_x + (r\dot{\theta})\,\widehat{\mathbf{b}}_y = r\dot{\theta}\,\widehat{\mathbf{b}}_y$$

$$^{N}\vec{\mathbf{a}}^Q = (0 - r\dot{\theta}^2)\,\widehat{\mathbf{b}}_x + (r\ddot{\theta} + 0)\,\widehat{\mathbf{b}}_y = -r\dot{\theta}^2\,\widehat{\mathbf{b}}_x + r\ddot{\theta}\,\widehat{\mathbf{b}}_y$$

Since we defined v as the particle's tangential velocity measure, we can relate that to ${}^{N}\vec{\mathbf{v}}^Q$ by dotting with the "tangential" unit vector $\widehat{\mathbf{u}}_t = \widehat{\mathbf{b}}_y$, and then also implicitly differentiating to get another relationship:

$$v \triangleq {}^{N}\vec{\mathbf{v}}^Q \cdot \widehat{\mathbf{b}}_y = r\dot{\theta}\,\widehat{\mathbf{b}}_y \cdot \widehat{\mathbf{b}}_y = r\dot{\theta}$$

$$v = r\dot{\theta}$$

$$\dot{v} = r\ddot{\theta}$$

Hence we can re-write ${}^{N}\vec{\mathbf{v}}^Q$ and ${}^{N}\vec{\mathbf{a}}^Q$ by plugging in $\dot{\theta} = v/r$ and $\ddot{\theta} = \dot{v}/r$ to get:

$$^{N}\vec{\mathbf{v}}^Q = v\,\widehat{\mathbf{b}}_y$$

$$^{N}\vec{\mathbf{a}}^Q = -(v^2/r)\,\widehat{\mathbf{b}}_x + \dot{v}\,\widehat{\mathbf{b}}_y$$

Finally, to resolve acceleration into normal-tangential measures, we can dot with ($\widehat{\mathbf{u}}_n = -\widehat{\mathbf{b}}_x$) and ($\widehat{\mathbf{u}}_t = \widehat{\mathbf{b}}_y$).

$$a_n = {}^{N}\vec{\mathbf{a}}^Q \cdot \widehat{\mathbf{u}}_n = v^2/r$$

$$a_t = {}^{N}\vec{\mathbf{a}}^Q \cdot \widehat{\mathbf{u}}_t = \dot{v}$$

It is **ok** to use these pre-computed expressions if you are **sure** that you have the signs correct. However, it is often convenient to assign frames in a different way (for example, in problem 7.7 frame B has $\widehat{\mathbf{u}}_n = -\widehat{\mathbf{b}}_x$ and $\widehat{\mathbf{u}}_t = \widehat{\mathbf{b}}_y$) so you cannot blindly use these expressions without accounting for this.

3.2 Example: Velocity and Acceleration of a Wrecking Ball

The figure shows a crane cab A supporting a boom B that swings a wrecking ball C_o. Right-handed orthogonal unit vectors $(\widehat{\mathbf{n}}_x, \widehat{\mathbf{n}}_y, \widehat{\mathbf{n}}_z)$, $(\widehat{\mathbf{b}}_x, \widehat{\mathbf{b}}_y, \widehat{\mathbf{b}}_z)$, and $(\widehat{\mathbf{c}}_x, \widehat{\mathbf{c}}_y, \widehat{\mathbf{c}}_z)$ are fixed in N, B, and C respectively, as shown.

The position vector from N_o to C_o is

$$^{N_o}\vec{\mathbf{r}}^{C_o} = x\,\widehat{\mathbf{n}}_x + L_B\,\widehat{\mathbf{b}}_x - L_C\,\widehat{\mathbf{c}}_y$$

Quantity	Symbol	Type
$\widehat{\mathbf{b}}_x$ distance between A_B and B_C	L_B	Constant
$\widehat{\mathbf{c}}_y$ distance between B_C and C_o	L_C	Constant
$\widehat{\mathbf{n}}_x$ distance between N_o to A_B	x	Variable
angle between $\widehat{\mathbf{n}}_x$ and $\widehat{\mathbf{b}}_x$ about $\widehat{\mathbf{n}}_z$	θ_B	Variable
angle between $\widehat{\mathbf{n}}_y$ and $\widehat{\mathbf{c}}_y$ about $\widehat{\mathbf{n}}_z$	θ_C	Variable
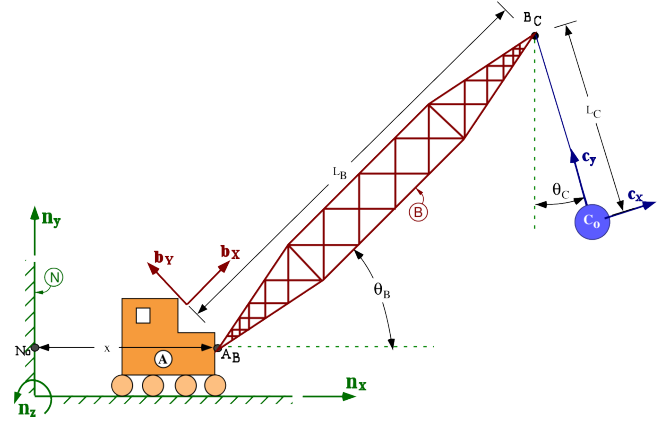
1. Determine:
    - the angular velocity of B in N (or, B's angular velocity in N).
    - the angular velocity of C in N (or, C's angular velocity in N).
    - the angular velocity of C in B (or, C's angular velocity in B).

> The first two are "simple" rotations (see section 7.3.3), so angular velocity is formed **by inspection**. For example, imagine θ_B is zero, and align your (right-hand!) fingers with $\widehat{\mathbf{b}}_x = \widehat{\mathbf{n}}_x$. Then mentally increase θ_B and curl your fingers in the direction of rotation. Your thumb gives the direction vector associated with $^N\vec{\boldsymbol{\omega}}^B$.
>
> $$^N\vec{\boldsymbol{\omega}}^B = \dot{\theta}_B\,\widehat{\mathbf{n}}_z \qquad\qquad ^N\vec{\boldsymbol{\omega}}^C = \dot{\theta}_C\,\widehat{\mathbf{n}}_z$$
>
> For $^B\vec{\boldsymbol{\omega}}^C$ it is easiest to use the angular velocity addition theorem (see section 7.3.4 - 7.3.5).
>
> $$^B\vec{\boldsymbol{\omega}}^C = {}^B\vec{\boldsymbol{\omega}}^N + {}^N\vec{\boldsymbol{\omega}}^C = (-\dot{\theta}_B + \dot{\theta}_C)\,\widehat{\mathbf{n}}_z$$

2. Compute the velocity of C_o in N in terms of symbols in the table and their time derivatives.

$$^N\vec{\mathbf{v}}^{C_o} = \boxed{\dot{x}\,\widehat{\mathbf{n}}_x + L_B\dot{\theta}_B\widehat{\mathbf{b}}_y + L_C\dot{\theta}_C\widehat{\mathbf{c}}_x}$$

> $$^N\vec{\mathbf{v}}^{C_o} \triangleq \frac{^Nd}{dt}\,{}^{N_o}\vec{\mathbf{r}}^{C_o} = \frac{^Nd}{dt}(x\,\widehat{\mathbf{n}}_x + L_B\,\widehat{\mathbf{b}}_x - L_C\,\widehat{\mathbf{c}}_y)$$
>
> We distribute first to get separate chunks to which we can then apply the golden rule:
>
> $$= \frac{^Nd}{dt}(x\,\widehat{\mathbf{n}}_x) + \qquad\qquad \frac{^Nd}{dt}(L_B\,\widehat{\mathbf{b}}_x) \qquad\qquad - \qquad\qquad \frac{^Nd}{dt}(L_C\,\widehat{\mathbf{c}}_y)$$
>
> $$= \frac{^Nd}{dt}(x\,\widehat{\mathbf{n}}_x) + [\,\frac{^Bd}{dt}(L_B\,\widehat{\mathbf{b}}_x) + {}^N\vec{\boldsymbol{\omega}}^B \times (L_B\,\widehat{\mathbf{b}}_x)\,] - [\,\frac{^Cd}{dt}(L_C\,\widehat{\mathbf{c}}_y) + {}^N\vec{\boldsymbol{\omega}}^C \times (L_C\,\widehat{\mathbf{c}}_y)\,]$$
>
> $$= \frac{^Nd}{dt}(x\,\widehat{\mathbf{n}}_x) + [\,\frac{^Bd}{dt}(L_B\,\widehat{\mathbf{b}}_x) + (\dot{\theta}_B\,\widehat{\mathbf{n}}_z) \times (L_B\,\widehat{\mathbf{b}}_x)\,] - [\,\frac{^Cd}{dt}(L_C\,\widehat{\mathbf{c}}_y) + (\dot{\theta}_C\,\widehat{\mathbf{n}}_z) \times (L_C\,\widehat{\mathbf{c}}_y)\,]$$
>
> $$= \dot{x}\,\widehat{\mathbf{n}}_x + [\,0 + L_B\dot{\theta}_B(\widehat{\mathbf{b}}_y)\,] - [\,0 + L_C\dot{\theta}_C(-\widehat{\mathbf{c}}_x)\,]$$
>
> $$= \dot{x}\,\widehat{\mathbf{n}}_x + L_B\dot{\theta}_B\widehat{\mathbf{b}}_y + L_C\dot{\theta}_C\widehat{\mathbf{c}}_x$$
>
> **Intuition Check**
>
> Take a moment to mentally visualize the movement of C_o caused by each "degree-of-freedom" (loosely speaking: each variable), and check that the terms of $^N\vec{\mathbf{v}}^{C_o}$ seem to describe that motion.
>
> **Checking Consistency of the Result**
>
> Vectors (like anything) must have the same units in order to add, so this is an easy way to check the result. It's a velocity, so each term must have the dimensions of length per time. In SI units, the first term is $\frac{m}{s}$ and the second and third terms are $m * \frac{rad}{s}$, which is equivalent to $\frac{m}{s}$ since radians are "unit-less".

3. Compute the acceleration of C_o in N. If it helps, you can use ${}^N\vec{\mathbf{a}}^{Bc} = \ddot{x}\,\widehat{\mathbf{n}}_{\mathrm{x}} + L_B\ddot{\theta}_B\,\widehat{\mathbf{b}}_{\mathrm{y}} - L_B(\dot{\theta}_B)^2\,\widehat{\mathbf{b}}_{\mathrm{x}}$.

$${}^N\vec{\mathbf{a}}^{C_o} \;=\; \boxed{\ddot{x}\,\widehat{\mathbf{n}}_{\mathrm{x}} \;+\; L_B\ddot{\theta}_B\,\widehat{\mathbf{b}}_{\mathrm{y}} \;-\; L_B(\dot{\theta}_B)^2\,\widehat{\mathbf{b}}_{\mathrm{x}} \;+\; L_C\ddot{\theta}_C\,\widehat{\mathbf{c}}_{\mathrm{x}} \;+\; L_C(\dot{\theta}_C)^2\,\widehat{\mathbf{c}}_{\mathrm{y}}}$$

Solution 1: Differentiation of Velocity

Differentiating the result from the previous part:

$$
\begin{aligned}
{}^{N}\vec{\mathbf{a}}^{C_o} &\triangleq \frac{{}^{N}d}{dt}\,{}^{N}\vec{\mathbf{v}}^{C_o} \\
&= \frac{{}^{N}d}{dt}(\dot{x}\,\widehat{\mathbf{n}}_{\mathrm{x}} + L_B\dot{\theta}_B\,\widehat{\mathbf{b}}_{\mathrm{y}} + L_C\dot{\theta}_C\,\widehat{\mathbf{c}}_{\mathrm{x}}) \\
&= \frac{{}^{N}d}{dt}(\dot{x}\,\widehat{\mathbf{n}}_{\mathrm{x}}) + \frac{{}^{N}d}{dt}(L_B\dot{\theta}_B\,\widehat{\mathbf{b}}_{\mathrm{y}}) + \frac{{}^{N}d}{dt}(L_C\dot{\theta}_C\,\widehat{\mathbf{c}}_{\mathrm{x}}) \\
&= \frac{{}^{N}d}{dt}(\dot{x}\,\widehat{\mathbf{n}}_{\mathrm{x}}) + [\,\frac{{}^{B}d}{dt}(L_B\dot{\theta}_B\,\widehat{\mathbf{b}}_{\mathrm{y}}) + {}^{N}\vec{\boldsymbol{\omega}}^{B}\times(L_B\dot{\theta}_B\,\widehat{\mathbf{b}}_{\mathrm{y}})\,] + [\,\frac{{}^{c}d}{dt}(L_C\dot{\theta}_C\,\widehat{\mathbf{c}}_{\mathrm{x}}) + {}^{N}\vec{\boldsymbol{\omega}}^{C}\times(L_C\dot{\theta}_C\,\widehat{\mathbf{c}}_{\mathrm{x}})\,] \\
&= \frac{{}^{N}d}{dt}(\dot{x}\,\widehat{\mathbf{n}}_{\mathrm{x}}) + [\,\frac{{}^{B}d}{dt}(L_B\dot{\theta}_B\,\widehat{\mathbf{b}}_{\mathrm{y}}) + {}^{N}\vec{\boldsymbol{\omega}}^{B}\times(L_B\dot{\theta}_B\,\widehat{\mathbf{b}}_{\mathrm{y}})\,] + [\,\frac{{}^{c}d}{dt}(L_C\dot{\theta}_C\,\widehat{\mathbf{c}}_{\mathrm{x}}) + {}^{N}\vec{\boldsymbol{\omega}}^{C}\times(L_C\dot{\theta}_C\,\widehat{\mathbf{c}}_{\mathrm{x}})\,] \\
&= \ddot{x}\,\widehat{\mathbf{n}}_{\mathrm{x}} + L_B\ddot{\theta}_B\,\widehat{\mathbf{b}}_{\mathrm{y}} - L_B(\dot{\theta}_B)^2\,\widehat{\mathbf{b}}_{\mathrm{x}} + L_C\ddot{\theta}_C\,\widehat{\mathbf{c}}_{\mathrm{x}} + L_C(\dot{\theta}_C)^2\,\widehat{\mathbf{c}}_{\mathrm{y}}
\end{aligned}
$$

Solution 2: Using Formulas for Two Points Fixed in a Rigid Frame

Since B_C and C_o are both **fixed** in frame C, we can use the formulas from Mitiguy section 8.2. Verify that you understand how the general formula was adapted to the specific frames and points in **this** problem to obtain the equation below.

$$
{}^{N}\vec{\mathbf{a}}^{C_o} = {}^{N}\vec{\mathbf{a}}^{B_c} + {}^{N}\vec{\boldsymbol{\alpha}}^{C}\times{}^{B_C}\vec{\mathbf{r}}^{C_o} + {}^{N}\vec{\boldsymbol{\omega}}^{C}\times({}^{N}\vec{\boldsymbol{\omega}}^{C}\times{}^{B_C}\vec{\mathbf{r}}^{C_o})
$$

We'll form each term separately:

- The first term $({}^{N}\vec{\mathbf{a}}^{B_c})$ was given.

- For the second term we need ${}^{N}\vec{\boldsymbol{\alpha}}^{C} \triangleq \frac{{}^{N}d}{dt}{}^{N}\vec{\boldsymbol{\omega}}^{C} = \ddot{\theta}_C\,\widehat{\mathbf{n}}_{\mathrm{z}}$, so then we can calculate:

$$
{}^{N}\vec{\boldsymbol{\alpha}}^{C}\times{}^{B_C}\vec{\mathbf{r}}^{C_o} = \ddot{\theta}_C\,\widehat{\mathbf{n}}_{\mathrm{z}}\times(-L_C\widehat{\mathbf{c}}_{\mathrm{y}}) = L_C\ddot{\theta}_C\,\widehat{\mathbf{c}}_{\mathrm{x}}
$$

- The third term is:

$$
{}^{N}\vec{\boldsymbol{\omega}}^{C}\times({}^{N}\vec{\boldsymbol{\omega}}^{C}\times{}^{B_C}\vec{\mathbf{r}}^{C_o}) = \dot{\theta}_C\,\widehat{\mathbf{n}}_{\mathrm{z}}\times(\dot{\theta}_C\,\widehat{\mathbf{n}}_{\mathrm{z}}\times(-L_C\widehat{\mathbf{c}}_{\mathrm{y}})) = \dot{\theta}_C\,\widehat{\mathbf{n}}_{\mathrm{z}}\times(L_C\dot{\theta}_C\,\widehat{\mathbf{c}}_{\mathrm{x}}) = L_C(\dot{\theta}_C)^2\,\widehat{\mathbf{c}}_{\mathrm{y}}
$$

Putting all the pieces together we have:

$$
{}^{N}\vec{\mathbf{a}}^{C_o} = \ddot{x}\,\widehat{\mathbf{n}}_{\mathrm{x}} + L_B\ddot{\theta}_B\,\widehat{\mathbf{b}}_{\mathrm{y}} - L_B(\dot{\theta}_B)^2\,\widehat{\mathbf{b}}_{\mathrm{x}} + L_C\ddot{\theta}_C\,\widehat{\mathbf{c}}_{\mathrm{x}} + L_C(\dot{\theta}_C)^2\,\widehat{\mathbf{c}}_{\mathrm{y}}
$$

Checking Consistency of the Result

Vectors (like anything) must have the same units in order to add, so this is an easy way to check the result.

- $\ddot{x} = \frac{d}{dt}(\frac{d}{dt}x)$ has dimensions of ((length/time)/time). In SI units, that would be m/s^2.

- L_B and L_C are both lengths (m), and $\ddot{\theta}_B$ and $\ddot{\theta}_C$ are both the second derivative of an angle (rad/s^2). Radians are "unit-less", so $L_B\ddot{\theta}_B$ and $L_C\ddot{\theta}_C$ both have units of m/s^2.

- $\dot{\theta}_B$ and $\dot{\theta}_C$ are first derivatives of an angle (rad/s), so the units on $L_B(\dot{\theta}_B)^2$ and $L_C(\dot{\theta}_C)^2$ are both m/s^2.

Finally, let's identify each term (see section 8.5) and check that it is consistent with the picture.

- The first term says that changes in x cause C_o to accelerate in the $\widehat{\mathbf{n}}_{\mathrm{x}}$ direction.

- The second term says there is an acceleration "tangent" to the rotation described by θ_B.

- The third term is due to the first-derivative of θ_B (i.e. due to B's angular velocity in N). Notice that it acts "inward" toward the source of the rotation; this is a "centripetal" term.

- The fourth term says there is an acceleration "tangent" to the rotation described by θ_C.

- The fifth term is due to the first-derivative of θ_C (i.e. due to C's angular velocity in N). Notice that it acts "inward" toward the source of the rotation; this is another "centripetal" term.

3.3 Example: Dynamics of a Rider on a Ferris Wheel

The figure at right shows a Ferris wheel. We take the ground to be a New-tonian reference frame N. Right-handed, orthogonal unit-vectors $\widehat{\mathbf{n}}_x, \widehat{\mathbf{n}}_y, \widehat{\mathbf{n}}_z$ are fixed in N as shown ($\widehat{\mathbf{n}}_x$ is out of the page).

The Ferris wheel B spins **counter-clockwise** at a variable rate ω and has chairs located at a radius R from the center point B_o. The rider sits on (and is fully supported by) a bathroom scale. We simplify the problem by modeling the rider as a particle Q and assuming that the seat is always exactly horizontal.

Hint: The concept of a rigid frame B is needed, but unit vectors fixed on B are **not** needed. Instead, use formulas for two points fixed on a rigid body.
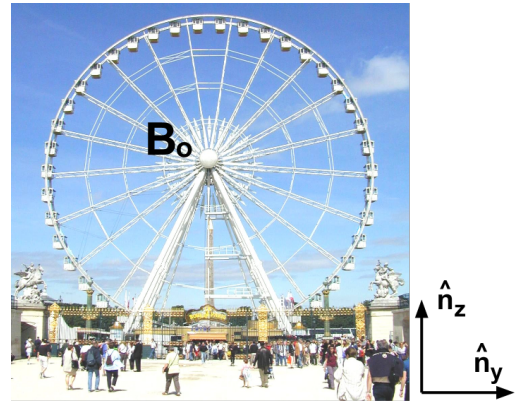
1. Write down the angular velocity and angular acceleration of B in N.

$$^{N}\vec{\boldsymbol{\omega}}^{B} = \boxed{\omega\,\widehat{\mathbf{n}}_x} \qquad\qquad {^{N}\vec{\boldsymbol{\alpha}}^{B}} = \boxed{\dot\omega\,\widehat{\mathbf{n}}_x}$$

> The Ferris wheel is undergoing "simple" rotation (see section 7.3.3), so angular velocity is formed **by inspection**. The rotation rate is given above as the variable ω, and the direction is picked by curling the fingers of your right hand to follow the counter-clockwise rotation, resulting in your thumb pointing out of the page in the $+\widehat{\mathbf{n}}_x$ direction.
>
> For angular acceleration we use the definition, $^{N}\vec{\boldsymbol{\alpha}}^{B} \triangleq \dfrac{^{N}d}{dt}{^{N}\vec{\boldsymbol{\omega}}^{B}}$, which is straight-forward since $\widehat{\mathbf{n}}_x$ is fixed in N.

2. Find an expression for the rider's acceleration in N **at the instant** when he is at the **top** of the Ferris wheel, in terms of previously defined symbols and $\widehat{\mathbf{n}}_x, \widehat{\mathbf{n}}_y, \widehat{\mathbf{n}}_z$. **Hint:** What is $^{B_o}\vec{\mathbf{r}}^{Q}$ at that instant?

$$^{N}\vec{\mathbf{a}}^{Q} = \boxed{-\,\dot\omega R\,\widehat{\mathbf{n}}_y - \omega^2 R\,\widehat{\mathbf{n}}_z}$$

> Point B_o is fixed in N, so we might be tempted to form the position vector, $^{B_o}\vec{\mathbf{r}}^{Q} = R\,\widehat{\mathbf{n}}_z$, and differentiate. However, this vector is only valid at the **instant** Q is at the top of the wheel (a short time before or after this instant, the direction from B_o to Q would include some $\widehat{\mathbf{n}}_y$ component), so it is **invalid** to differentiate it. (As we have done in other problems, one approach would be to introduce a basis fixed on B so that $^{B_o}\vec{\mathbf{r}}^{Q}$ could be written in a form that is valid for all time.)
>
> Instead, since B_o and Q are both fixed on B, the equations for velocity and acceleration of two points fixed on a rigid body apply (Mitiguy section 8.2).
>
> $$\begin{aligned} ^{N}\vec{\mathbf{a}}^{Q} &= {^{N}\vec{\mathbf{a}}^{B_o}} + {^{N}\vec{\boldsymbol{\alpha}}^{B}} \times {^{B_o}\vec{\mathbf{r}}^{Q}} + {^{N}\vec{\boldsymbol{\omega}}^{B}} \times ({^{N}\vec{\boldsymbol{\omega}}^{B}} \times {^{B_o}\vec{\mathbf{r}}^{Q}}) && \text{General equation valid for all time.}\\ &= \vec{\mathbf{0}} + \dot\omega\,\widehat{\mathbf{n}}_x \times R\,\widehat{\mathbf{n}}_z + \omega\,\widehat{\mathbf{n}}_x \times (\omega\,\widehat{\mathbf{n}}_x \times R\,\widehat{\mathbf{n}}_z) && \text{Valid at this instant due to expression for } {^{B_o}\vec{\mathbf{r}}^{Q}}.\\ &= \vec{\mathbf{0}} + \dot\omega R\,(-\widehat{\mathbf{n}}_y) + \omega\,\widehat{\mathbf{n}}_x \times (\omega R(-\widehat{\mathbf{n}}_y))\\ &= \vec{\mathbf{0}} + \dot\omega R\,(-\widehat{\mathbf{n}}_y) + \omega^2 R\,(-\widehat{\mathbf{n}}_z) \end{aligned}$$

3. The forces on the rider might be modeled as: a horizontal friction force F_y and vertical support force F_z from the scale; and gravity. Hence, $\vec{\mathbf{F}}^{Q} = F_y\,\widehat{\mathbf{n}}_y + (F_z - m^Q g)\,\widehat{\mathbf{n}}_z$. Plug $\vec{\mathbf{F}}^{Q}$ and your result for $^{N}\vec{\mathbf{a}}^{Q}$ into Newton's law to get a vector equation, then dot with a unit vector to get a scalar equation relating F_z and ω.

> $$\begin{aligned} \vec{\mathbf{F}}^{Q} &= m^Q * {^{N}\vec{\mathbf{a}}^{Q}}\\ F_y\,\widehat{\mathbf{n}}_y + (F_z - m^Q g)\,\widehat{\mathbf{n}}_z &= m^Q * (-\dot\omega R\,\widehat{\mathbf{n}}_y - \omega^2 R\,\widehat{\mathbf{n}}_z) && F_z \text{ is in an } \widehat{\mathbf{n}}_z \text{ term, so we'll dot with } \widehat{\mathbf{n}}_z.\\ F_z - m^Q g &= -m^Q * \omega^2 R && \text{This could be solved for any scalar of interest.} \end{aligned}$$

3.4 Example: Acceleration of a Car (2D)

A car (rigid frame C) is traveling on a flat road (assumed to be a Newtonian frame N). Frames N and C have right-handed, orthogonal unit vectors $\hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_y$, $\hat{\mathbf{n}}_z$ and $\hat{\mathbf{c}}_x$, $\hat{\mathbf{c}}_y$, $\hat{\mathbf{c}}_z$ fixed in N and C, respectively.

- $\hat{\mathbf{n}}_x$ points east, $\hat{\mathbf{n}}_y$ points north, $\hat{\mathbf{n}}_z$ points "up" away from the ground.
- $\hat{\mathbf{c}}_x$ points "forward" in the car, $\hat{\mathbf{c}}_y$ points "left" from the car, and $\hat{\mathbf{c}}_z = \hat{\mathbf{n}}_z$.
- You are intentionally not given the angle between $\hat{\mathbf{n}}_x$ and $\hat{\mathbf{c}}_x$; with the golden rule you don't need it.

Since we have assumed the car rotates only about ($\hat{\mathbf{n}}_z = \hat{\mathbf{c}}_z$), its angular velocity in N could be described by one variable, $^N\vec{\boldsymbol{\omega}}^C = \omega_z\,\hat{\mathbf{c}}_z$. The car tires roll forward and can slide sideways, so the velocity in N of the car's center of mass can be described by two parameters, $^N\vec{\mathbf{v}}^{C_{cm}} = v_x\,\hat{\mathbf{c}}_x + v_y\,\hat{\mathbf{c}}_y$. Quantities v_x, v_y, and ω_z may vary with time.

1. Differentiate $^N\vec{\mathbf{v}}^{C_{cm}}$ in N to get the acceleration of the car's center of mass in N.

$$^N\vec{\mathbf{a}}^{C_{cm}} \triangleq \frac{^Nd\,^N\vec{\mathbf{v}}^{C_{cm}}}{dt} = \boxed{(\dot{v}_x - \omega_z v_y)\,\hat{\mathbf{c}}_x + (\dot{v}_y + \omega_z v_x)\,\hat{\mathbf{c}}_y}$$

All components are expressed in C, so we don't need to distribute. We'll just apply the golden rule:

$$= \frac{^Cd}{dt}(v_x\,\hat{\mathbf{c}}_x + v_y\,\hat{\mathbf{c}}_y) + \quad ^N\vec{\boldsymbol{\omega}}^C \times (v_x\,\hat{\mathbf{c}}_x + v_y\,\hat{\mathbf{c}}_y)$$

$$= \frac{^Cd}{dt}(v_x\,\hat{\mathbf{c}}_x + v_y\,\hat{\mathbf{c}}_y) + (\omega_z\,\hat{\mathbf{c}}_z) \times (v_x\,\hat{\mathbf{c}}_x + v_y\,\hat{\mathbf{c}}_y)$$

$$= [\,\dot{v}_x\,\hat{\mathbf{c}}_x + \dot{v}_y\,\hat{\mathbf{c}}_y\,] + [\,\omega_z v_x\,\hat{\mathbf{c}}_y + \omega_z v_y(-\hat{\mathbf{c}}_x)\,]$$

$$= (\dot{v}_x - \omega_z v_y)\,\hat{\mathbf{c}}_x + \quad\quad (\dot{v}_y + \omega_z v_x)\,\hat{\mathbf{c}}_y$$

Take a moment to mentally picture what the presence of each variable tells you about the acceleration.

2. At a certain instant, the car's speedometer measures 60 mph, the tires are not sliding, and the gyro measures ω_z as 0.5 rad/sec. Calculate and sketch the radius of curvature of C_{cm}'s path at this instant.

Since the tires are **not** sliding, the non-zero value of ω_z implies the car is driving around a curve. We make use of a little geometry to see the relationship between speed along the arc, angular velocity, and radius of curvature.

$\rho = \frac{v_x \Delta t}{\omega_z \Delta t} = \frac{60 \text{ mph}}{0.5 \text{ rad/s}} = \frac{26.82 \text{ m/s}}{0.5 \text{ rad/s}} = 53.6$ meters.
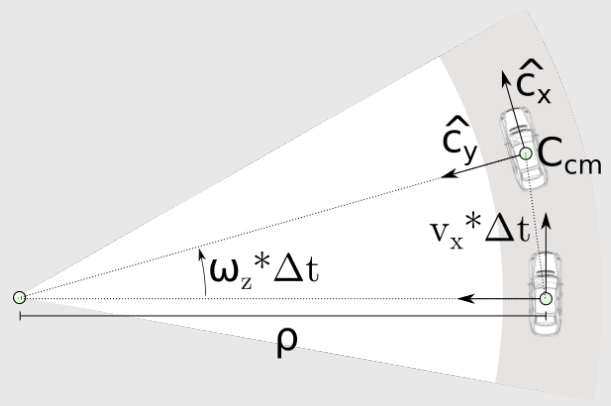
3. Now assume the car travels at a constant forward speed ($\dot{v}_x = 0$) and does not slide ($v_y = \dot{v}_y = 0$). Using your result from (a), does constant speed imply zero acceleration?

$$^N\vec{\mathbf{a}}^{C_{cm}} = (0 - \omega_z * 0)\,\hat{\mathbf{c}}_x + (0 + \omega_z v_x)\,\hat{\mathbf{c}}_y$$

$$= 0\,\hat{\mathbf{c}}_x + \omega_z v_x\,\hat{\mathbf{c}}_y$$

What does this result say? It says the C_{cm} has non-zero acceleration as long as there is simultaneously a non-zero speed and a non-zero rate of rotation, **even if they are both constant (not changing)**.

Note that if we do the substitution $w_z = v_x/\rho$ based on the scenario in (b), then we get

$$^N\vec{\mathbf{a}}^{C_{cm}} = (\frac{v_x}{\rho})v_x\,\hat{\mathbf{c}}_y = \frac{v_x^2}{\rho}\,\hat{\mathbf{c}}_y$$

Dotting with $\hat{\mathbf{c}}_y$ gives the magnitude of "centripetal" acceleration, which will be familiar from introductory physics.

Chapter 4

Kinematic Constraints

4.1 Common Constraints

There are some common constraints that arise in "typical" engineering problems. The Mitiguy book describes constraints in abstract, but this handout attempts to give an intuition for some specific examples of constraints.

4.1.1 Specified Relationships

When one scalar a is a **known** function of another scalar b, **substitution** is used to eliminate one of the variables. **Implicit differentiation** can be used to relate the derivatives, for example $\dot{r} = f(\dot{\theta})$ and $\ddot{r} = g(\ddot{\theta})$, in order to eliminate r, $\dot{r}$, and $\ddot{r}$.

Some common examples include:

- cam-followers, where the cam profile is given as an explicit relationship between r and θ.
- rigid slots, where the profile is known function of a variable.

Examples: (borrowed from Hibbeler 12th Edition)

F12–36. Peg P is driven by the forked link OA along the path described by $r = e^{\theta}$. When $\theta = \frac{\pi}{4}$ rad, the link has an angular velocity and angular acceleration of $\dot{\theta} = 2$ rad/s and $\ddot{\theta} = 4$ rad/s². Determine the radial and transverse components of the peg's acceleration at this instant.
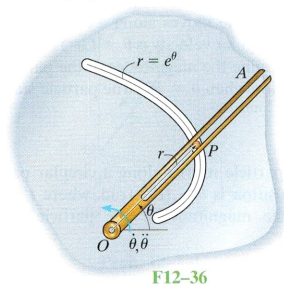
F12–37. The collars are pin-connected at B and are free to move along rod OA and the curved guide OC having the shape of a cardioid, $r = [0.2(1 + \cos\theta)]$ m. At $\theta = 30°$, the angular velocity of OA is $\dot{\theta} = 3$ rad/s. Determine the magnitudes of the velocity of the collars at this point.
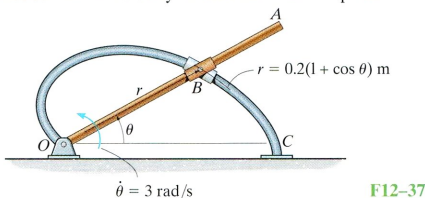
F12–38. At the instant $\theta = 45°$, the athlete is running with a constant speed of 2 m/s. Determine the angular velocity at which the camera must turn in order to follow the motion.

F12–36

$\dot{\theta} = 3$ rad/s

F12–37

$r = (30 \csc\theta)$ m

30 m

F12–38
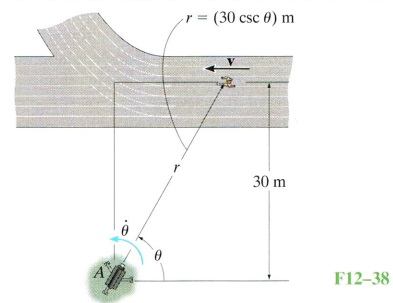
4.1.2 Pin or hinge constraints

A pin or hinge dictates that the two points on two separate bodies (or frames) that are pinned together must have the same position, velocity, and acceleration. Hence, the constraints are:

$$^{A_B}\vec{\mathbf{r}}^{B_A} = \vec{\mathbf{0}}$$

$$^{\text{AnyFrame}}\vec{\mathbf{v}}^{A_B} = {}^{\text{AnyFrame}}\vec{\mathbf{v}}^{B_A}$$

$$^{\text{AnyFrame}}\vec{\mathbf{a}}^{A_B} = {}^{\text{AnyFrame}}\vec{\mathbf{a}}^{B_A}$$

Example: Mitiguy Homework Problem 4.16 shows a position constraint.

4.1.3 Rolling

When body B **rolls** on body A, the **instantaneous** contact points must have the same velocity (but NOT the same acceleration!). Hence, the the motion constraint is:

$$^{\text{AnyFrame}}\vec{\mathbf{v}}^{A_B} = {}^{\text{AnyFrame}}\vec{\mathbf{v}}^{B_A}$$

Examples: All of Mitiguy Homework Set 8.

**Examples:** (borrowed from Hibbeler 12th Edition)

16–6. The mechanism for a car window winder is shown in the figure. Here the handle turns the small cog C, which rotates the spur gear S, thereby rotating the fixed-connected lever AB which raises track D in which the window rests. The window is free to slide on the track. If the handle is wound at 0.5 rad/s, determine the speed of points A and E and the speed v_w of the window at the instant $\theta = 30°$.
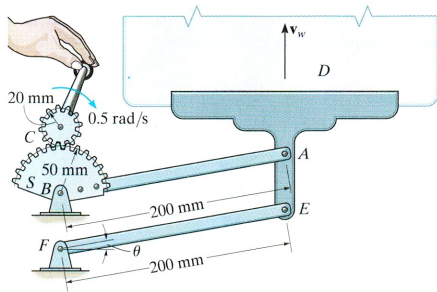
•16–9. When only two gears are in mesh, the driving gear A and the driven gear B will always turn in opposite directions. In order to get them to turn in the same direction an idler gear C is used. In the case shown determine the angular velocity of gear B when $t = 5$ s, if gear A starts from rest and has an angular acceleration of $\alpha_A = (3t + 2)$ rad/s^2, where t is in seconds.
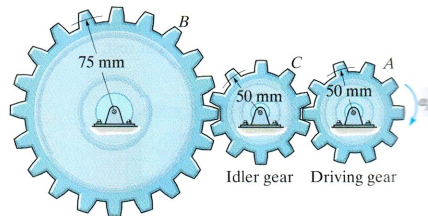
*16–80. If the ring gear D rotates counterclockwise with an angular velocity of $\omega_D = 5$ rad/s while link AB rotates clockwise with an angular velocity of $\omega_{AB} = 10$ rad/s, determine the angular velocity of gear C.
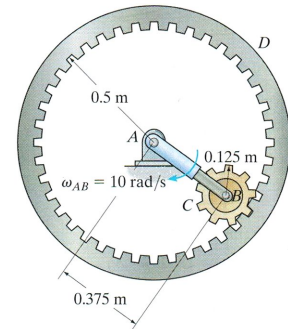

4.1.4 Slots

In a mechanism with slots, the physical insight is that the slot places a constraint on motion **perpendicular** to the slot. That is, the point **in** the slot and the adjacent point **of** the slot must have the same velocity and acceleration **in the direction** $\widehat{\mathbf{u}}_\perp$ **perpendicular to the slot at that instant**. Hence the constraints are:

$$\left(^{\text{AnyFrame}}\vec{\mathbf{v}}^{A_B}\right) \cdot \widehat{\mathbf{u}}_\perp = \left(^{\text{AnyFrame}}\vec{\mathbf{v}}^{B_A}\right) \cdot \widehat{\mathbf{u}}_\perp$$

$$\left(^{\text{AnyFrame}}\vec{\mathbf{a}}^{A_B}\right) \cdot \widehat{\mathbf{u}}_\perp = \left(^{\text{AnyFrame}}\vec{\mathbf{a}}^{B_A}\right) \cdot \widehat{\mathbf{u}}_\perp$$

Examples: (borrowed from Hibbeler 12th Edition)

•16–81. If the slider block A is moving to the right at $v_A = 8$ ft/s, determine the velocity of blocks B and C at the instant shown. Member CD is pin connected to member ADB.
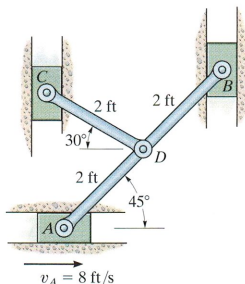
•16–145. The disk rolls without slipping and at a given instant has the angular motion shown. Determine the angular velocity and angular acceleration of the slotted link BC at this instant. The peg at A is fixed to the disk.
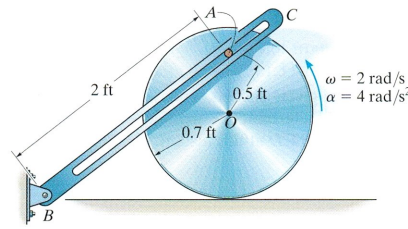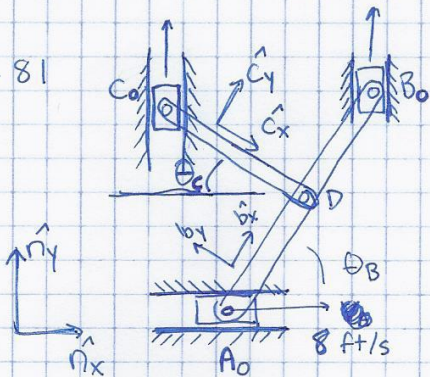
16–147. The two-link mechanism serves to amplify angular motion. Link AB has a pin at B which is confined to move within the slot of link CD. If at the instant shown, AB (input) has an angular velocity of $\omega_{AB} = 2.5$ rad/s and an angular acceleration of $\alpha_{AB} = 3$ rad/s^2, determine the angular velocity and angular acceleration of CD (output) at this instant.
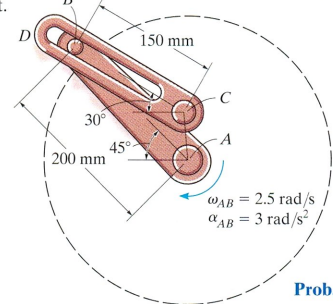
Prob. 16–145

Prob. 16–147

16-81

$^n R^b$	$\hat{b_x}$	$\hat{b_y}$	$\hat{b_z}$
$\hat{n_x}$	$\cos\theta_B$	$-\sin\theta_B$	0
$\hat{n_y}$	$\sin\theta_B$	$\cos\theta_B$	0
$\hat{n_z}$	0	0	1

$^n R^c$	$\hat{c_x}$	$\hat{c_y}$	$\hat{c_z}$
$\hat{n_x}$	$\cos\theta_c$	$\sin\theta_c$	0
$\hat{n_y}$	$-\sin\theta_c$	$\cos\theta_c$	0
$\hat{n_z}$	0	0	1

Given: $^N\vec{V}^{A_0} = V_A \hat{n_x} = 8 \text{ ft/s } \hat{n_x}$.   Let $L_B \triangleq |\vec{r}^{B_0/A_0}|$ and $L_C \triangleq |\vec{r}^{C_0/D}|$.

Assume $^N\vec{V}^{B_0} = V_B \hat{n_y}$ and $^N\vec{V}^C = V_C \hat{n_y}$

Two-points-fixed:

$$^N\vec{V}^{B_0} = {}^N\vec{V}^{A_0} + {}^N\vec{\omega}^B \times \vec{r}^{B_0/A_0}$$

① $V_B \hat{n_y} = V_A \hat{n_x} + (\dot{\theta}_B \hat{n_z}) \times (L_B \hat{b_x}) = V_A \hat{n_x} + L_B \dot{\theta}_B \hat{b_y}$

dot w/ $\hat{b_x}$; so ~~that~~ that we can get rid of unknown $\underline{\dot{\theta}_B}$.

$\hookrightarrow$ $V_B \sin\theta_B = V_A \cos\theta_B$

$\rightarrow V_B = V_A \frac{1}{\tan\theta_B} = (8 \frac{ft}{s})(1) = 8 \text{ ft/s}$

Now for D: $^N\vec{V}^D = {}^N\vec{V}^{A_0} + {}^N\vec{\omega}^B \times \vec{r}^{D/A_0}$

② $V_x \hat{n_x} + V_y \hat{n_y} = V_A \hat{n_x} + (\dot{\theta}_B \hat{n_z}) \times (~~\text{} ~~ L_B/2 \, \hat{b_x})$
                           $= V_A \hat{n_x} + \frac{1}{2} L_B \dot{\theta}_B \hat{b_y}$

Still don't know $\dot{\theta}_B$, so we'll dot ① with $\hat{b_y}$ now to solve:

① dot w/ $\hat{b_y}$: $V_B \cos\theta_B = V_A (-\sin\theta_B) + L_B \dot{\theta}_B \rightarrow \dot{\theta}_B = \frac{V_B \cos\theta_B + V_A \sin\theta_B}{L_B}$

Now use ②: $= 2.82 \text{ rad/s}$

dot ② w/ $\hat{n_x}$: $V_x = V_A + \frac{1}{2} L_B \dot{\theta}_B (-\sin\theta_B) \rightarrow V_x = 4 \text{ ft/s}$

dot ② w/ $\hat{n_y}$: $V_y = \frac{1}{2} L_B \dot{\theta}_B \cos\theta_B \longrightarrow V_y = 4 \text{ ft/s}$

Now for C_0: $^N\vec{V}^{C_0} = {}^N\vec{V}^D + {}^N\vec{\omega}^C \times \vec{r}^{C_0/D}$

$V_C \hat{n_y} = (V_x \hat{n_x} + V_y \hat{n_y}) + (-\dot{\theta}_c \hat{n_z}) \times (-L_c \hat{c_x})$
              $= V_x \hat{n_x} + V_y \hat{n_y} + L_c \dot{\theta}_c \hat{c_y}$

Don't know $\dot{\theta}_c$, so dot w/ $\hat{c_x}$:

$V_C (-\sin\theta_c) = V_x \cos\theta_c + V_y (-\sin\theta_c) \longrightarrow V_C = -2.93 \text{ ft/s}$

②

$$\hat{a_y} \cdot \hat{b_x} = \cos(90 - \theta_B + \theta_A)$$
$$= \cos(90 - (\theta_B - \theta_A))$$
$$= \sin(\theta_B - \theta_A)$$

$^nR^a$	$\hat{a_x}$	$\hat{a_y}$	$\hat{a_z}$
$\hat{n_x}$	$\cos\theta_A$	$-\sin\theta_A$	0
$\hat{n_y}$	$\sin\theta_A$	$\cos\theta_A$	0
$\hat{n_z}$	0	0	1

$^nR^b$	$\hat{b_x}$	$\hat{b_y}$	
$\hat{n_x}$	$\cos\theta_B$	$-\sin\theta_B$	0
$\hat{n_y}$	$\sin\theta_B$	$\cos\theta_B$	0
$\hat{n_z}$	0	0	1

$$^N\vec{V}^{AB} = {}^N\vec{V}^{AN} + {}^N\vec{\omega}^A \times \vec{r}^{AB/AN}$$

$$= \vec{0} + (\dot{\theta}_A \hat{a_z}) \times (R\hat{n_y} + r\hat{a_x})$$

$$= -\dot{\theta}_A R \hat{n_x} + \dot{\theta}_A r \hat{a_y}$$

$$^N\vec{V}^{BA} = {}^N\vec{V}^{Bo} + {}^N\vec{\omega}^B \times \vec{r}^{BA/Bo}$$

$$= \vec{0} + (\dot{\theta}_B \hat{b_z}) \times (L_B \hat{b_x})$$

$$= L_B \dot{\theta}_B \hat{b_y}$$

Slot constraint: $\left({}^N\vec{V}^{AB} \cdot \hat{b_y}\right) = \left({}^N\vec{V}^{BA} \cdot \hat{b_y}\right)$

$$+ R\dot{\theta}_A \sin\theta_B + r\dot{\theta}_A \cos(\theta_A - \theta_B) = L_B \dot{\theta}_B \quad ①$$

Note: $\omega_B = \dot{\theta}_B$, $\omega_A = \dot{\theta}_A$

Implicit diff:

$$R\ddot{\theta}_A \sin\theta_B + R\dot{\theta}_A \dot{\theta}_B \cos\theta_B + r\ddot{\theta}_A \cos(\theta_A - \theta_B) - r\dot{\theta}_A(\dot{\theta}_A - \dot{\theta}_B)\sin(\theta_A - \theta_B) = \dot{L}_B \dot{\theta}_B \quad ②$$
$$+ L_B \ddot{\theta}_B$$

$$\dot{L}_B = {}^N\vec{V}^{AB} \cdot \hat{b_x} = (-\dot{\theta}_A R \hat{n_x} \cdot \hat{b_x} + \dot{\theta}_A r \hat{a_y} \cdot \hat{b_x})$$

$$= -\dot{\theta}_A R \cos\theta_B + \dot{\theta}_A r \sin(\theta_B - \theta_A) \quad ③$$

Note: $\ddot{\theta}_A = \alpha_A \qquad \ddot{\theta}_B = \alpha_B$

Use equation 1 to solve for $\omega_B = \dot{\theta}_B$.

Use equation 2 (and plug-in $\dot{L}_B$ from ③) to solve for $\alpha_B = \ddot{\theta}_B$.

4.2 Example: Kinematic Analysis for a Slider-Crank Mechanism

The following two problems are typical EIT-style slider-crank problems. These problems are sometimes categorized as "relative motion analysis" and/or "instant-center" methods in traditional textbooks. The purpose of this handout is to demonstrate how to solve this variety of planar kinematics problems without resorting to graphical instant-center methods. The solutions to the two problems could probably be combined and generalized to a set of formulas based on the lengths of the links and configuration angles, but that would defeat the purpose of knowing how to do general kinematics!

4.2.1 Relating Velocity and Angular Velocity

The figure below shows a slider-crank mechanism. If member AB has a constant angular velocity of 3 radians/sec clockwise, the velocity of slider C at the instant shown is:

a. 12 in/sec

b. 9 in/sec

c. 16 in/sec

d. 15 in/sec

e. zero (it is momentarily at rest)
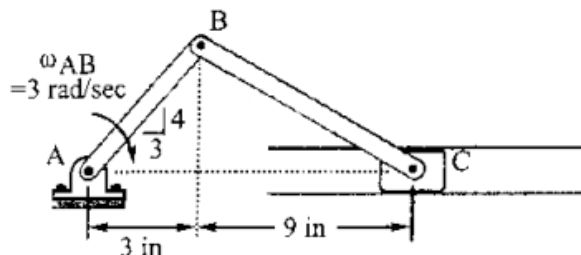
Solution: The traditional approach to this problem involves sketching on the picture to graphically find a point that is the instant center of rotation for member BC. However, we can solve it in a more methodical way by considering the velocity equation for two points fixed on a rigid body.

Let's re-label the drawing using our notation style. Link A is connected to N by a hinge at A_o and to link B by a hinge at A_B. Link C is modeled as a particle, and is connected to B by a hinge. As suggested by the picture, C is constrained to slide in the slot horizontally.
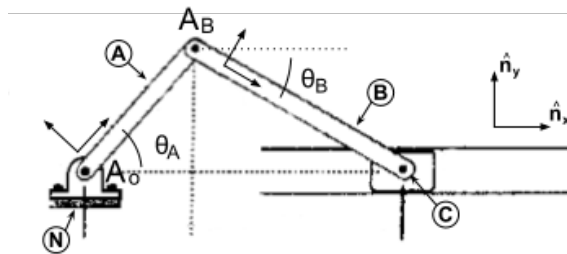
For convenience, we define our usual right-handed, orthogonal unit vectors fixed in N with $\widehat{\mathbf{n}}_x$ horizontally right and $\widehat{\mathbf{n}}_y$ vertically upward. Let A have similar unit vectors with $\widehat{\mathbf{a}}_z = \widehat{\mathbf{n}}_z$ and $\widehat{\mathbf{a}}_x$ pointing from A_o to A_B, and likewise B has $\widehat{\mathbf{b}}_z = \widehat{\mathbf{n}}_z$ and $\widehat{\mathbf{b}}_x$ pointing from A_B to C. Then we can write $^N\vec{\boldsymbol{\omega}}^A = -\omega_A\,\widehat{\mathbf{n}}_z$ and $^N\vec{\boldsymbol{\omega}}^B = -\omega_B\,\widehat{\mathbf{n}}_z$.

Our desired unknown is the measure of C's velocity in the $\widehat{\mathbf{n}}_x$ direction. To get there, we need to know something about the motion of link B, which means we need to know the velocity of A_B. Fortunately, since we know the motion of link A we can directly compute the velocity of point A_B using the equation for two points fixed on a rigid body:

$$^N\vec{\mathbf{v}}^{A_B} = {}^N\vec{\mathbf{v}}^{A_o} + {}^N\vec{\boldsymbol{\omega}}^A \times {}^{A_o}\vec{\mathbf{r}}^{A_B} = \vec{\mathbf{0}} + (-\omega_A\,\widehat{\mathbf{n}}_z) \times (L_A\,\widehat{\mathbf{a}}_x)$$

And similarly we can relate the velocity of point A_B and point C.

$$^N\vec{\mathbf{v}}^C = {}^N\vec{\mathbf{v}}^{A_B} + {}^N\vec{\boldsymbol{\omega}}^B \times {}^{A_B}\vec{\mathbf{r}}^C$$

$$v_x\,\widehat{\mathbf{n}}_x + v_y\,\widehat{\mathbf{n}}_y = (-\omega_A\,\widehat{\mathbf{n}}_z) \times (L_A\,\widehat{\mathbf{a}}_x) + (-\omega_B\,\widehat{\mathbf{n}}_z) \times (L_B\,\widehat{\mathbf{b}}_x)$$

$$v_x\,\widehat{\mathbf{n}}_x + v_y\,\widehat{\mathbf{n}}_y = (-\omega_A\,L_A)\,\widehat{\mathbf{a}}_y + (-\omega_B\,L_B)\,\widehat{\mathbf{b}}_y$$

In the above equation there are 3 unknowns (v_y, v_x, ω_B). We can get 2 scalar equations by dotting with $\widehat{\mathbf{n}}_y$ and $\widehat{\mathbf{n}}_x$, and a third equation by considering the slot constraint, arguing that it enforces $v_y = 0$.

$$v_y = 0$$

$$v_y = (-\omega_A\,L_A)\,\widehat{\mathbf{a}}_y \cdot \widehat{\mathbf{n}}_y + (-\omega_B\,L_B)\,\widehat{\mathbf{b}}_y \cdot \widehat{\mathbf{n}}_y$$

$$v_x = (-\omega_A\,L_A)\,\widehat{\mathbf{a}}_y \cdot \widehat{\mathbf{n}}_x + (-\omega_B\,L_B)\,\widehat{\mathbf{b}}_y \cdot \widehat{\mathbf{n}}_x$$

The lengths are calculated from the picture:

$$L_A = \sqrt{3^2 + 4^2} = 5 \qquad\qquad\qquad L_B = \sqrt{4^2 + 9^2} = 9.85$$

Finally, the simple rotation matrices $^{\mathrm{N}}R^{\mathrm{A}}$ and $^{\mathrm{N}}R^{\mathrm{B}}$ take the form:

$^{\mathrm{N}}R^{\mathrm{A}}$	$\widehat{\mathbf{a}}_{\mathrm{x}}$	$\widehat{\mathbf{a}}_{\mathrm{y}}$	$\widehat{\mathbf{a}}_{\mathrm{z}}$
$\widehat{\mathbf{n}}_{\mathrm{x}}$	$\cos(\theta_A)$	$-\sin(\theta_A)$	0
$\widehat{\mathbf{n}}_{\mathrm{y}}$	$\sin(\theta_A)$	$\cos(\theta_A)$	0
$\widehat{\mathbf{n}}_{\mathrm{z}}$	0	0	1

$^{\mathrm{B}}R^{\mathrm{N}}$	$\widehat{\mathbf{n}}_{\mathrm{x}}$	$\widehat{\mathbf{n}}_{\mathrm{y}}$	$\widehat{\mathbf{n}}_{\mathrm{z}}$
$\widehat{\mathbf{b}}_{\mathrm{x}}$	$\cos(\theta_B)$	$-\sin(\theta_B)$	0
$\widehat{\mathbf{b}}_{\mathrm{y}}$	$\sin(\theta_B)$	$\cos(\theta_B)$	0
$\widehat{\mathbf{b}}_{\mathrm{z}}$	0	0	1

which can be filled in either by using trig to compute $\theta_A = 53.13°$ and $\theta_B = 23.96°$, or by using "*soh cah toa*" to replace the trig functions by the appropriate ratios:

$^{\mathrm{n}}R^{\mathrm{a}}$	$\widehat{\mathbf{a}}_{\mathrm{x}}$	$\widehat{\mathbf{a}}_{\mathrm{y}}$	$\widehat{\mathbf{a}}_{\mathrm{z}}$
$\widehat{\mathbf{n}}_{\mathrm{x}}$	3/5	$-4/5$	0
$\widehat{\mathbf{n}}_{\mathrm{y}}$	4/5	3/5	0
$\widehat{\mathbf{n}}_{\mathrm{z}}$	0	0	1

$^{\mathrm{b}}R^{\mathrm{n}}$	$\widehat{\mathbf{n}}_{\mathrm{x}}$	$\widehat{\mathbf{n}}_{\mathrm{y}}$	$\widehat{\mathbf{n}}_{\mathrm{z}}$
$\widehat{\mathbf{b}}_{\mathrm{x}}$	9/9.85	$-4/9.85$	0
$\widehat{\mathbf{b}}_{\mathrm{y}}$	4/9.85	9/9.85	0
$\widehat{\mathbf{b}}_{\mathrm{z}}$	0	0	1

So our system of equations reduces to:

$$0 = (-\omega_A\, L_A)\, \cos(\theta_A) + (-\omega_B\, L_B)\, (\cos(\theta_B))$$

$$v_x = (-\omega_A\, L_A)\, (-\sin(\theta_A)) + (-\omega_B\, L_B)\, (\sin(\theta_B))$$

which is solved as:

$$w_B = (-\omega_A\, \frac{L_A}{L_B})\, (\frac{3}{5})\, (\frac{9.85}{9}) = -3\, \frac{\mathrm{rad}}{\mathrm{s}} * \frac{5\ \mathrm{in}}{9.85\ \mathrm{in}} * \frac{3}{5} * \frac{9.85}{9} = -1\, \frac{\mathrm{rad}}{\mathrm{s}}$$

$$v_x = (-3\, \frac{\mathrm{rad}}{\mathrm{s}})\, (5\ \mathrm{in})\, (-\frac{4}{5}) + (1\, \frac{\mathrm{rad}}{\mathrm{s}})\, (9.85\ \mathrm{in})\, (\frac{4}{9.85}) = 16\frac{\mathrm{in}}{\mathrm{s}}$$

It seems our result matches choice (c) in the question, which is indeed the correct answer.

It is worth noting that this analysis is quite general; similar slider-crank problems can be solved by computing the lengths and rotation tables given whatever new pictorial geometry is presented.

4.2.2 Relating Acceleration and Angular Acceleration

Below is shown a mechanism consisting of a rotating disk AB, a link BC, and a slider at C. The wheel AB has a constant angular velocity of 6 radians/sec. At the instant shown, the link BC is translating (its angular velocity is zero). The angular acceleration of link BC is:

a. 0

b. 15 rad/sec² CCW

c. 9 rad/sec² CW

d. 6 rad/sec² CCW

e. 13 rad/sec² CW
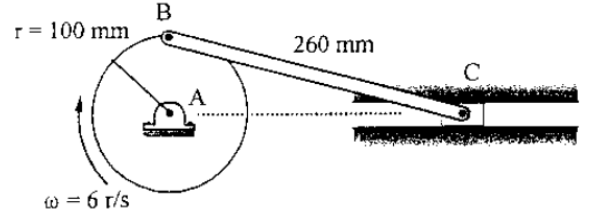
Solution:

We will once again use the two-points-fixed equations to analyze the kinematics, although this time we will need to use the acceleration equation as well as the velocity equation.

Let's re-label the drawing using our notation style. The configuration of the mechanism has been changed slightly to better show the angles and basis vectors we've assigned. Not shown on the drawing are r_A, the radius of the wheel, and L_B, the length of link B.
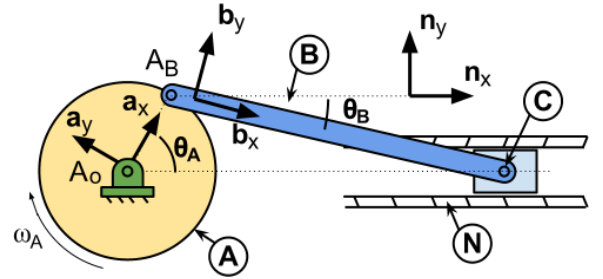
For convenience, we define our usual right-handed, orthogonal unit vectors fixed in N with $\widehat{\mathbf{n}}_x$ horizontally right and $\widehat{\mathbf{n}}_y$ vertically upward. Let A have similar unit vectors with $\widehat{\mathbf{a}}_z = \widehat{\mathbf{n}}_z$ and $\widehat{\mathbf{a}}_x$ pointing from A_o to A_B, and likewise B has $\widehat{\mathbf{b}}_z = \widehat{\mathbf{n}}_z$ and $\widehat{\mathbf{b}}_x$ pointing from A_B to C.

Going with the clockwise convention suggested in the picture, we express the angular velocities and angular accelerations as:

$$^N\vec{\boldsymbol{\omega}}^A = -\omega_A\,\widehat{\mathbf{a}}_z \qquad\qquad\qquad ^N\vec{\boldsymbol{\omega}}^B = -\omega_B\,\widehat{\mathbf{b}}_z$$
$$^N\vec{\boldsymbol{\alpha}}^A = -\dot{\omega}_A\,\widehat{\mathbf{a}}_z \qquad\qquad\qquad ^N\vec{\boldsymbol{\alpha}}^B = -\dot{\omega}_B\,\widehat{\mathbf{b}}_z$$

As before, the simple rotation matrices $^NR^A$ and $^NR^B$ take the form

$^NR^A$	$\widehat{\mathbf{a}}_x$	$\widehat{\mathbf{a}}_y$	$\widehat{\mathbf{a}}_z$
$\widehat{\mathbf{n}}_x$	$\cos(\theta_A)$	$-\sin(\theta_A)$	0
$\widehat{\mathbf{n}}_y$	$\sin(\theta_A)$	$\cos(\theta_A)$	0
$\widehat{\mathbf{n}}_z$	0	0	1

$^BR^N$	$\widehat{\mathbf{n}}_x$	$\widehat{\mathbf{n}}_y$	$\widehat{\mathbf{n}}_z$
$\widehat{\mathbf{b}}_x$	$\cos(\theta_B)$	$-\sin(\theta_B)$	0
$\widehat{\mathbf{b}}_y$	$\sin(\theta_B)$	$\cos(\theta_B)$	0
$\widehat{\mathbf{b}}_z$	0	0	1

where $\theta_A = 90°$ and $\theta_B = 22.62°$ at the instant considered in the problem (found using trig).

For link B, we can relate its angular acceleration to the acceleration of the end-points by:

$$^N\vec{\mathbf{a}}^C \;=\; ^N\vec{\mathbf{a}}^{A_B} \;+\; ^N\vec{\boldsymbol{\alpha}}^B \times {}^{A_B}\vec{\mathbf{r}}^C \;+\; ^N\vec{\boldsymbol{\omega}}^B \times \left(^N\vec{\boldsymbol{\omega}}^B \times {}^{A_B}\vec{\mathbf{r}}^C \right) \tag{4.1}$$

which suggests that we need to find the acceleration of A_B and of C, and the angular velocity of B.

We can get the acceleration of A_B by:

$$^N\vec{\mathbf{a}}^{A_B} \;=\; ^N\vec{\mathbf{a}}^{A_o} \;+\; ^N\vec{\boldsymbol{\alpha}}^A \times {}^{A_o}\vec{\mathbf{r}}^{A_B} \;+\; ^N\vec{\boldsymbol{\omega}}^A \times \left(^N\vec{\boldsymbol{\omega}}^A \times {}^{A_o}\vec{\mathbf{r}}^{A_B} \right)$$

$$^N\vec{\mathbf{a}}^{A_B} \;=\; \vec{\mathbf{0}} \;+\; (-\dot{\omega}_A\,\widehat{\mathbf{a}}_z) \times (r_A\,\widehat{\mathbf{a}}_x) \;+\; -\omega_A\,\widehat{\mathbf{a}}_z \times (-\omega_A\,\widehat{\mathbf{a}}_z \times r_A\,\widehat{\mathbf{a}}_x)$$

$$^N\vec{\mathbf{a}}^{A_B} \;=\; -r_A\dot{\omega}_A\,\widehat{\mathbf{a}}_y \;-\; r_A\omega_A^2\,\widehat{\mathbf{a}}_x \tag{4.2}$$

Plugging equation (4.2) into (4.1), and assuming that $^N\vec{\mathbf{a}}^C$ is of the form $^N\vec{\mathbf{a}}^C = a_x\,\widehat{\mathbf{n}}_x + 0\,\widehat{\mathbf{n}}_y$ (by physical reasoning, the slot constraint enforces that there is only a horizontal component) we get:

$$a_x\,\widehat{\mathbf{n}}_x \;+\; 0\,\widehat{\mathbf{n}}_y \;=\; -r_A\dot{\omega}_A\,\widehat{\mathbf{a}}_y \;-\; r_A\omega_A^2\,\widehat{\mathbf{a}}_x \;+\; (-\dot{\omega}_B\,\widehat{\mathbf{b}}_z) \times L_B\,\widehat{\mathbf{b}}_x \;+\; (-\omega_B\,\widehat{\mathbf{b}}_z) \times (-\omega_B\,\widehat{\mathbf{b}}_z \times L_B\,\widehat{\mathbf{b}}_x)$$

$$a_x \, \widehat{\mathbf{n}}_{\mathrm{x}} \; + \; 0 \, \widehat{\mathbf{n}}_{\mathrm{y}} \;=\; -\, r_A \dot{\omega}_A \, \widehat{\mathbf{a}}_{\mathrm{y}} \;-\; r_A \omega_A^2 \, \widehat{\mathbf{a}}_{\mathrm{x}} \;-\; \dot{\omega}_B L_B \, \widehat{\mathbf{b}}_{\mathrm{y}} \;-\; \omega_B^2 L_B \, \widehat{\mathbf{b}}_{\mathrm{x}} \tag{4.3}$$

Note that, so far, we haven't assumed a constant rate of rotation of A. Now we can simplify our expression by plugging in $\dot{\omega}_A = 0$ and $\omega_B = 0$ as stated in the problem:

$$a_x \, \widehat{\mathbf{n}}_{\mathrm{x}} \;=\; -\, r_A \omega_A^2 \, \widehat{\mathbf{a}}_{\mathrm{x}} \;-\; \dot{\omega}_B L_B \, \widehat{\mathbf{b}}_{\mathrm{y}} \tag{4.4}$$

We are left with two unknowns. Dotting equation 4.4 with any two independent unit vectors will give us two independent scalar equations; let's pick $\widehat{\mathbf{n}}_{\mathrm{x}}$ and $\widehat{\mathbf{n}}_{\mathrm{y}}$ for simple look-up in the rotation tables:

dot w/ $\widehat{\mathbf{n}}_{\mathrm{x}}$:	$a_x \;=\; -\, r_A \omega_A^2 \cos(\theta_A) \;-\; \dot{\omega}_B L_B \sin(\theta_B)$
dot w/ $\widehat{\mathbf{n}}_{\mathrm{y}}$:	$0 \;=\; -\, r_A \omega_A^2 \sin(\theta_A) \;-\; \dot{\omega}_B L_B \cos(\theta_B)$

Plugging in $\theta_A = 90°$, $\theta_B = 22.62°$, $\omega_A = 6$ rad/s, $r_A = 100$ mm, and $L_B = 260$ mm, we find:

$$\dot{\omega}_B \;=\; -\,15 \ \mathrm{rad/s^2}$$

$$a_x \;=\; 1500 \ \mathrm{mm/s^2} \;=\; 1.5 \ \mathrm{m/s^2}$$

And finally we find that answer (b) is the correct choice:

$$^{N}\vec{\boldsymbol{\alpha}}^{B} = -\dot{\omega}_B \, \widehat{\mathbf{b}}_{\mathrm{z}} = +15 \ \mathrm{rad/s^2} \ \widehat{\mathbf{b}}_{\mathrm{z}} = 15 \ \mathrm{rad/s^2} \ \mathrm{CCW}$$

Addendum:
The problem tells us $\omega_B = 0$ at the instant shown, but let's prove it using the two-points-fixed velocity equation:

$$^{N}\vec{\mathbf{v}}^{A_B} \;=\; ^{N}\vec{\mathbf{v}}^{A_o} + {}^{N}\vec{\boldsymbol{\omega}}^{A} \times {}^{A_o}\vec{\mathbf{r}}^{A_B} \;=\; \vec{\mathbf{0}} + (-\omega_A \, \widehat{\mathbf{n}}_{\mathrm{z}}) \times (L_A \, \widehat{\mathbf{a}}_{\mathrm{x}})$$

which we can plug into:

$$^{N}\vec{\mathbf{v}}^{C} \;=\; ^{N}\vec{\mathbf{v}}^{A_B} + {}^{N}\vec{\boldsymbol{\omega}}^{B} \times {}^{A_B}\vec{\mathbf{r}}^{C}$$

$$v_x \, \widehat{\mathbf{n}}_{\mathrm{x}} + 0 \, \widehat{\mathbf{n}}_{\mathrm{y}} \;=\; (-\omega_A \, \widehat{\mathbf{n}}_{\mathrm{z}}) \times (L_A \, \widehat{\mathbf{a}}_{\mathrm{x}}) + (-\omega_B \, \widehat{\mathbf{n}}_{\mathrm{z}}) \times (L_B \, \widehat{\mathbf{b}}_{\mathrm{x}})$$

$$v_x \, \widehat{\mathbf{n}}_{\mathrm{x}} \;=\; (-\omega_A \, L_A) \, \widehat{\mathbf{a}}_{\mathrm{y}} \;+\; (-\omega_B \, L_B) \, \widehat{\mathbf{b}}_{\mathrm{y}} \tag{4.5}$$

Dotting equation (4.5) with any two independent unit vectors lets one solve for the unknowns v_x and w_B:

dot w/ $\widehat{\mathbf{n}}_{\mathrm{x}}$:	$v_x \;=\; \omega_A L_A \sin(\theta_A) \;-\; \omega_B L_B \sin(\theta_B)$
dot w/ $\widehat{\mathbf{n}}_{\mathrm{y}}$:	$0 \;=\; \omega_A L_A \cos(\theta_A) \;-\; \omega_B L_B \cos(\theta_B)$

The result is $\omega_B = 0$ when $\theta_A = 90°$.